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Research Article

Hyperideal Structure of Krasner's Induced Quotient Hypperings

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Abstract: This paper mainly explores the hyperideal structure of Krasner's induced quotient hyperrings. By Krasner's induced hyperring, we mean an additive hyperring R/G induced on a ring R by one of its multiplicative subgroups G. In 1983, Krasner introduced a way of constructing this class of hyperrings and posed the question of whether all the hyperrings that exist naturally were either isomorphic to or could be embedded into this kind of derived hyperrings. Later, in 1985, Massouros proposed a method of construction of hyperfields, which are not embeddable in any of Krasner's construction. Subsequently, the focus on this important class of additive hyperrings has subsided. We revive the interest in this class of hyperrings by investigating the relationships between the ideals of R and the hyperideals of R/G attentively. Also, we introduce a way of constructing additive hyperrings from given additive hyperrings, and consequently, we prove a few theorems that exhibit the isomorphic relationship between this class of hyperrings under certain conditions.

Keywords: Hyperrings, Hyperideals, Krasner's Hyperrings, Krasner's induced quotient Hyperrings.

1. Introduction

Fundamentally, hyperstructures, as opposed to classical algebraic structures, possess one or more hyperoperations that satisfy a certain set of axioms.

Marty (1934) introduced the idea of hyperstructures. He introduced the notion of hyperoperation and thereby defined the structure called hypergroups. Krasner (1953) introduced the notion of hyperrings, as hyperstructure analogue of rings, by having hyperaddition in place of addition in the rings and satisfying certain axioms similar to that of a ring, as a tool on the approximation of valued fields. These structures are currently referred to as additive hyperrings or Krasner's hyperrings. Later, other variations of hyperrings, such as multiplicative hyperrings and general hyperrings, were introduced by mathematicians. Also, other hyperstructures like hypermodules, hypersemigroups, topological hypergroups, polygroups, and hypervectorspaces were also introduced and studied systematically by several mathematicians (Davvaz and Leoreanu-Fotea, 2007; Davvaz, 2020).

Krasner (1983) introduced a way of constructing hyperrings and hyperfields from given rings and fields, respectively, as a quotient structure induced by a multiplicative subgroup of the given ring or field, and subsequently posed a question of whether all the hyperrings were isomorphic to, or embeddable in, a hyperring of this class. Later, Massouros (1985) proposed a method of constructing hyperfields that are neither isomorphic to, nor embeddable in, a hyperring of this class. After this was settled, interest in this special class of induced quotient hyperrings has waned largely.

By Krasner's induced quotient hyperring, we mean an additive hyperring induced on a ring by one of its multiplicative subgroups or any additive hyperring which is isomorphic to such one. The Krasner's method of construction (Krasner, 1983) of such hyperrings R/G from rings R is as follows: Let R be any ring and G be a multiplicative subgroup of R. Define $R/G = \{aG \mid a \in R\}$. and the multiplication " $\cdot$ ", hyperaddition " $+$ " , on R/G as $aG \cdot bG = abG$. and $aG + bG = \{tG \mid t = ag_1 + bg_2;$ for some $g_1, g_2 \in G\}$.

Later, various other ways of constructing hyperrings were introduced by several mathematicians (Stefanescu, 2006), and many such methods were all compiled in this expository paper. Recently, Ameri et al. (2021) proposed a method of constructing a general hyperring from a ring by a modification of Krasner's original method.

Hyperstructures, in general, and hyperrings, in particular, have many applications in a variety of fields such as automata theory, cryptography, chemical reactions (Davvaz and Leoreanu-Fotea, 2007), and other branches of mathematics (Davvaz, 2020). In this paper, we study the structural properties of various classes of hyperideals, such as maximal, prime, and primary hyperideals, and prove some results that are similar to the isomorphism theorems of the ring theory. We believe that our work will be of immense use once the classification of finite hyperrings formally begins by mathematicians in the future.

2. Preliminaries

Here, we compile certain important definitions that we will use in the next section. Also, all the hyperrings concerned, in this section and the subsequent ones, are commutative additive hyperrings with identity 1.

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1. **Hyperoperation.** A hyperoperation defined on a non-empty set R is a map from $R \times R$ to the set of all non-empty subsets of R.
That is, if $a, b \in R$ and $+$ is any hyperoperation on R, then $a + b \subseteq R \setminus \{\emptyset\}$.

2. **Additive hyperring / Krasner's hyperring.**
A non-empty set R with a hyperaddition "$+$" and multiplication "$\cdot$" is called an additive hyperring if for every $a, b, c \in R$,
   (a) $(a + b) + c = a + (b + c)$,
   (b) $a + b = b + a$,
   (c) $\exists\, 0 \in R$ s.t $a + 0 = \{a\} = 0 + a$,
   (d) $\forall a \in R,\ \exists - a \in R$ s.t $0 \in a + (-a)$,
   (e) If $a \in b + c$ then $b \in a + (-c),\ c \in a + (-b)$,
   (f) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
   (g) $a \cdot 0 = 0 = 0 \cdot a$,
   (h) $a \cdot (b + c) = a \cdot b + a \cdot c,\ (a + b) \cdot c = a \cdot c + b \cdot c$.

   In addition, most additive hyperrings have the multiplicative identity 1 satisfying the following: $a1 = 1a = a$ for each a. Also, if R has no zero divisors, then we call R to be a hyperdomain, and if every non-zero element of R is invertible, then we call R to be a hyperfield.

3. **Krasner's induced quotient hyperring.**
Let R be a ring and G be a multiplicative subgroup of R. Then the left cosets of G in R, denoted by R/G, forms an additive hyperring under the following operations:
$aG + bG = \{tG \mid t = ag_1 + bg_2 : \text{for some } g_1, g_2 \in G\}$ and $aG \cdot bG = abG$.
We call such hyperrings or hyperrings that are isomorphic to such hyperrings as Krasner's induced quotient hyperrings.

4. **Hyperideal.**
A non-empty subset I of R is called a hyperideal of given hyperring R, if, for $a, b \in I$ and $r \in R$
   (a) $a - b \subseteq I$,
   (b) $ra, ar \in I$.

5. **Maximal hyperideal.**
A hyperideal M of a hyperring R is called maximal, if $M \subsetneq J \subseteq R$ and J is a hyperideal of R then $J = R$.

6. **Prime hyperideal.**
A hyperideal P of a hyperring R is called prime, if $ab \in P$ then $a \in P$ or $b \in P$.

7. **Primary hyperideal**
A hyperideal Q of R is called primary, if $ab \in Q$ and $a \notin Q$ then $\exists n \in \mathbb{N}$ such that $b^n \in Q$.

8. **Hyperring homomorphism.**
Let R, S be commutative hyperrings with 1. We say that the function $f : R \longrightarrow S$ is a homomorphism, if, for each $a, b \in R$,
   (a) $f(a + b) = f(a) + f(b)$,
   (b) $f(ab) = f(a) f(b)$,
   (c) $f(1_R) = 1_S$.

   In addition, if f is a bijection, we say that f is an isomorphism, in which case we call R and S to be isomorphic.

9. **Noetherian Hyperring.**
A hyperring R is called Noetherian if it satisfies the ascending chain condition for hyperideals. That is, if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ is an ascending chain of hyperideals in R, then there exists $m \in \mathbb{N}$ such that $I_n = I_m$ for each $n \geq m$.

10. **Artinian Hyperring.** A hyperring R is called Artinian if it satisfies the descending chain condition for hyperideals. That is, if $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ is a descending chain of hyperideals in R, then there exists $m \in \mathbb{N}$ such that $I_n = I_m$ for each $n \geq m$.

3. Results and discussions

Throughout this section, we assume R to be a commutative ring with the identity 1 and G as a multiplicative subgroup of R with the identity e. Unless otherwise specified, we assume $1 \neq e$.

Proposition 3.1.

1. If R is an integral domain, then R/G is a hyperdomain,
2. If R/G is a hyperdomain, then R is an integral domain if and only if "e" is not a zero divisor.

Proof.

1. Suppose R is an integral domain.
If $aG, bG \in R/G$ such that $aGbG = 0G$ then $abG = 0G$. We have $abe = 0$. Since R has no zero-divisors, $ae = 0$ or $be = 0$, which implies $aG = 0G$ or $bG = 0G$.
Hence, R/G is an integral domain.

2. Suppose that R/G is an integral domain, and "e" is not a zero divisor of R.
   If $a, b \in R$ such that $ab = 0$, then $aGbG = 0G$, which implies $aG = 0G$ or $bG = 0G$, and hence $ae = 0$ or $be = 0$. Thus, $a = 0$ or $b = 0$, as "e" is a non-zero divisor. Therefore, R is an integral domain.
   Conversely, suppose that R is an integral domain. Since $e \in G$, $e \neq 0$, and so, we conclude that "e" is not a zero divisor.

□

Proposition 3.2. Let R be an integral domain. Then R/G is a hyperfield if and only if for each $a \in R \setminus \{0\}$, there exists $b \in R$ such that $ab \in G$.

Proof. Suppose that R/G is a hyperfield. Take $a \in R \setminus \{0\}$, and so $aG (\neq 0G) \in R/G$. Otherwise, $aG = 0G$ and so, $ae = 0$, which implies $a = 0$ as $e \neq 0$, a contradiction. Since R/G is a hyperfield, there exists $b^{'}G \in R/G$ such that $aGb^{'}G = 1G$. Now $ab^{'}G = 1G$, and so, $ab^{'}e = g$, for some $g \in G$. We rename $b^{'}e = b(\neq 0)$. Thus, we have $ab \in G$.
Conversely, suppose that, for each $a \in R \setminus \{0\}$ there exists $b \in R$ such that $ab \in G$.
Take $aG \in R/G$ with $aG \neq 0G$. So, clearly, we have $a \neq 0$, and therefore, there exists $b \in R$ such that $ab \in G$. So, $abG = G$. That is, $aGbG = 1G$.
Hence R/G is a hyperfield. □

Proposition 3.3. Let R and S be commutative rings with identity. Let G and H be multiplicative subgroups of R and S, respectively. Then,

1. $G \times H$ is a multiplicative subgroup of $R \times S$,
2. $(R \times S) / (G \times H) \cong R/G \times S/H$.

Proof.

1. This is obvious, as $G \times H \subseteq R \times S$ and $G \times H$ is a group under componentwise multiplication, equipped with $R \times S$.
2. Define $\Phi : (R \times S) / (G \times H) \longrightarrow R/G \times S/H$ by $\Phi((r, s) G \times H) = (rG, sH)$.
   We have $(r, s)(G \times H) = (r_1, s_1)(G \times H)$ if and only if $(r, s)(e_1, e_2) = (r_1, s_1)(g, h)$; for some $(g, h) \in G \times H$, here e_1, e_2 are identities of G and H, respectively. Which is equivalent to $re_1 = r_1 g$ and $se_2 = s_1 h$, for some $g \in G, h \in H$. And, in turn, this is equivalent to $rG = r_1 G$ and $sH = s_1 H$, if and only if $(rG, sH) = (r_1 G, s_1 H)$. Hence, we conclude Φ is both well-defined and one-to-one.
   Clearly, Φ is an onto function.
   If $\Phi((a, b)(G \times H)) \in \Phi((r, s)(G \times H) + (r_1, s_1)(G \times H))$ with $(a, b)(G \times H) \in (r, s)(G \times H) + (r_1, s_1)(G \times H)$.
   Then, $(a, b)(G \times H) = ((r, s)(g, h) + (r_1, s_1))(g_1, h_1)\ (G \times H)$;for some $g, g_1 \in G$ and $h, h_1 \in H$, and hence $(a, b)(e_1, e_2) = ((r, s)(g, h) + (r_1, s_1)(g_1, h_1))(x, y)$, where $(x, y) \in (G \times H)$. So, we have $ae_1 = rgx + r_1 g_1 x, be_2 = shy + s_1 h_1 y$.
   Hence, $aG = (rgx + r_1 g_1 x) G \in rG + r_1 G$, $bH = (shy + s_1 h_1 y) H \in sH + s_1 H$.
   So, $\Phi((a, b)(G \times H)) = (aG, bH) \in (rG + r_1 G, sH + s_1 H) = (rG, sH) + (r_1 G,\ s_1 H) = \Phi((r, s)(G \times H)) + \Phi((r_1, s_1)(G \times H))$.
   Thus we have
   $\Phi((r, s)(G \times H) + (r_1, s_1)(G \times H)) \subseteq \Phi((r, s)(G \times H)) + \Phi((r_1, s_1)(G \times H))$.
   Take$(aG, bH) \in \Phi((r, s)(G \times H)) + \Phi((r_1, s_1)(G \times H)) = (rG, sH) + (r_1 G, s_1 H) = (rG + r_1 G,\ sH + s_1 H)$. So, $aG \in rG + r_1 G, bH \in sH + s_1 H$.
   As a consequence, we have $a \cdot e = rg + r_1 g_1$, $b \cdot e = sh + s_1 h_1$; for some $g, g_1 \in G$, $h, h_1 \in H$.
   Hence, $aG = (rg + r_1 g_1) G$ and $bH = (sh + s_1 h_1) H$.
   $(aG, bH) = ((rg + r_1 g_1) G, (sh + s_1 h_1) H) = \Phi((rg + r_1 g_1), (sh + s_1 h_1)(G \times H))$.
   Since $(rg + r_1 g_1, sh + s_1 h_1)(G \times H) = ((r, s)(g, h) + (r_1, s_1)(g_1, h_1)) G \times H \in (r, s) G \times H + (r_1, s_1) G \times H$.
   That is,
   $(aG, bH) \in \Phi((r, s)(G \times H) + (r_1, s_1)(G \times H))$.
   Hence, $\Phi((r, s)(G \times H) + (r_1, s_1)(G \times H)) = \Phi((r, s)(G \times H)) + \Phi((r_1, s_1)(G \times H))$.
   Thus, Φ is an isomorphism.

□

Proposition 3.4. If I is an ideal of R, and G is a multiplicative subgroup of R, then $I/G = \{iG | i \in I\}$ forms a hyperideal of R/G.

Proof. Since $I \neq \varnothing$, we have $I/G \neq \varnothing$. Let $i_1 G, i_2 G \in I/G$.
Consider $i_1 G - i_2 G = i_1 G + (-i_2) G = \{tG |\ t = i_1 g_1 - i_2 g_2, \text{ for some } g_1, g_2 \in G\}$.
Since $t = i_1 g_1 - i_2 g_2 \in I$, we have $i_1 G - i_2 G \subseteq I/G$.
Also, if $rG \in R/G$, then $rG \cdot i_1 G = ri_1 G \in I/G$ as $ri_1 = i_1 r \in I$.
Hence I/G is a hyperideal of R/G □

Proposition 3.5. Let $aG, bG \in R/G$, then $aG = bG$ if and only if $ae = bg$, for some $g \in G$.

Proof. Suppose that $aG = bG$. So, $ae \in aG = bG$. And therefore, there exists $g \in G$ such that $ae = bg$.
Conversely, suppose that $ae = bg$, for some $g \in G$. If $x \in aG$, then $x = ag_1$, for some $g_1 \in G$. We have

$x = aeg_1 = bgg_1 \in bG$. Consequently $aG \subseteq bG$. As we have $be = ag^{-1}$, by a similar argument, we can prove $bG \subseteq aG$. Therefore $aG = bG$.

□

Proposition 3.6. Any hyperideal $I^{'}$ of R/G is of the form I/G, where I is an ideal of R.

Proof. Define $I = \left\{r \in R : rG \in I^{'}\right\}$. Since $0G \in I^{'}, 0 \in I$, and so, $I \neq \emptyset$ Take $i_1, i_2 \in I$,and $r \in R$. We have $(i_1 - i_2)\, G = (i_1 e - i_2 e)\, G \in i_1 G - i_2 G \subseteq I^{'}$, as $e \in G$. And so, $i_1 - i_2 \in I$. Also, $rG \cdot i_1 G = ri_1 G \in I^{'}$ as $i_1 G \in I^{'}$. Thus, $ri_1 = i_1 r \in I$. And therefore, I is an ideal of R. Now, we claim that $I/G = I^{'}$. To prove this, take $rG \in I^{'}$. So, we have $r \in I$. Thus, $rG \in I/G$. If $rG \in I/G$, then $rG = iG$, for some $i \in I$. Which implies $re = ig$, for some $g \in G$. As a result of which, we have $re \in I$. Therefore, $(re)\, G = rG \in I^{'}$. Lastly, we have $I^{'} = I/G$.

□

Proposition 3.7. If I and J are two ideals of R with $I/G = J/G$, then $I\{e\} = J\{e\}$.

Proof. Take $i \cdot e \in I\{e\}$ with $i \in I$.
Since $i \in I, iG \in I/G = J/G$, we have $iG = jG$, for some $j \in J$. So, there is $g \in G$ such that $ie = jg$. Therefore we have $ie \in J$. Now consider $i\,(ee) = ie \in J\{e\}$. And thus $I\{e\} \subseteq J\{e\}$. Similarly, we can prove $J\{e\} \subseteq I\{e\}$. Hence $I\{e\} = J\{e\}$. □

Remarks:-

1. Converse of the above proposition holds trivially as $I\{e\}/G = I/G$ and $J\{e\}/G = J/G$.
2. If $e = 1$, then $I/G = J/G$ if and only if $I = J$.

Proposition 3.8. If I and J are ideals of R,then

1. $(I \cap J)/G = (I/G) \cap (J/G)$,
2. $(I \cup J)/G = (I/G) \cup (J/G)$,
3. $(I + J)/G = (I/G) + (J/G)$,
4. $(I \cdot J)/G = (I/G) \cdot (J/G)$,
5. $\sqrt{I}/G = \sqrt{I/G}$.

Proof.

1. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, we have $(I \cap J)/G \subseteq (I/G) \cap (J/G)$.
If, on the other hand, $iG = jG$ with $i \in I, j \in J$, then $ie = jg$, for some $g \in G$.
Thus we have $ie \in I \cap J$. Now consider $iG = (ie)\, G \in (I \cap J)\{e\}/G = (I \cap J)/G$. Therefore $(I \cap J)/G = (I/G) \cap (J/G)$.

2. Clearly, we have $(I \cup J)/G \supseteq (I/G) \cup (J/G)$.
If $xG \in (I \cup J)/G$, then $xG = yG$, for some $y \in I \cup J$. Thus, we have $xG = yG$ with $y \in I$ or $y \in J$. And therefore $xG \in I/G$ or $xG \in J/G$, according as $y \in I$ or $y \in J$. Hence, we have $xG \in (I/G) \cup (J/G)$. Therefore, $(I \cup J)/G = (I/G) \cup (J/G)$.

3. Take $(i + j)/G \in (I + J)/G$ with $i \in I, j \in J$.
We have $(i + j)\, G = (i \cdot e + j \cdot e)\, G \in iG + jG \subseteq (I/G) + (J/G)$. And therefore, $(I + J)/G \subseteq (I/G) + (J/G)$. Now take $xG \in (I/G) + (J/G)$. Consequently, we have $xG \in (iG + jG)$ with $i \in I, j \in J$. Hence, there exist g_1, g_2 in G such that $xG = (ig_1 + jg_2)\, G$. So, $xe = (ig_1 + jg_2)\, g$, for some $g \in G$. Thus, by distributivity, $xe = ig_1 g + jg_2 g \in I + J$, which implies $xG = (xe)\, G \in (I + J)/G$.
Thus, $(I + J)/G = (I/G) + (J/G)$.

4. Take $(\sum_{k=1}^{n} i_k j_k)\, G \in (I \cdot J)/G$ with $i_k \in I, j_k \in J$. Then $(\sum_{k=1}^{n} i_k j_k)\, G = (\sum_{k=1}^{n} i_k j_k) \cdot e\ G = (\sum_{k=1}^{n} i_k j_k e)\, G \in \sum_{k=1}^{n} i_k j_k G = \sum_{k=1}^{n} i_k G j_k G \in (I/G) \cdot (J/G)$. If $xG \in (I/G) \cdot (J/G)$, then $xG \in \sum_{k=1}^{n} (a_k b_k G)$, where $a_k \in I, b_k \in J$. So, we have $xG = (\sum_{k=1}^{n} a_k b_k g_k)\, G \in IJ/G$ as $\sum_{k=1}^{n} a_k\,(b_k g_k) \in IJ$.
Thus, $(I \cdot J)/G = (I/G) \cdot (J/G)$.

5. If $xG \in \sqrt{I/G}$, then there exists $n \in \mathbb{N}$ such that $(xG)^n \in I/G$. So, we have $x^n G \in I/G$, and which implies that $x^n e = ig$, for some $i \in I, g \in G$. So, we have $x^n e \in I$. Now, consider $(xe)^n = x^n e \in I$. and hence $xe \in \sqrt{I}$. Therefore, $xG = xe\ G \in \sqrt{I}/G$. Conversely, suppose that $xG \in \sqrt{I}/G$ with $x \in \sqrt{I}$. As a result, we have an $n \in \mathbb{N}$ such that $x^n \in I$. Hence, $(xG)^n = x^n G \in I/G$. Thus, $xG \in \sqrt{I/G}$. So, we conclude that $\sqrt{I}/G = \sqrt{I/G}$.

□

Proposition 3.9. If $P^{'}$ is a prime hyperideal of R/G, then there is a prime ideal P of R such that $P^{'} = P/G$, provided $1 \in G$.

Proof. Since $P^{'}$ is a hyperideal of R/G, there is an ideal P of R such that $P^{'} = P/G$.
Now, it remains to show that P is a prime ideal of R. So, let $ab \in P$, then $abG \in P^{'}$. Since $P^{'}$ is a prime hyperideal of R, we have $aG \in P^{'}$ or $bG \in P^{'}$. If $aG \in P^{'} = P/G$, $a1 = pg$ for some $p \in P$ and $g \in G$.

As a result, $a \in P$. Similarly, we can prove that if $bG \in P^{'}$, then $b \in P$. So, we have either $a \in P$ or $b \in P$. Hence, P is a prime ideal, and $P^{'} = P/G$.

$\square$

Proposition 3.10. If Q is a primary ideal of R, then Q/G is a primary hyperideal of R/G.

Proof. Suppose that Q is a primary ideal. To prove Q/G is a primary ideal, let $aGbG \in Q/G$ with $aG \notin Q/G$. We have $abG \in Q/G$ with $aG \notin Q/G$, and which implies that $abe = qg$, for some $q \in Q$ and $g \in G$. Thus, we have $abe \in Q$. Since $aG \notin Q/G$, we have $a \notin Q$, and therefore, there exists an n in $\mathbb{N}$ such that $(be)^n = b^n e \in Q$. So, $b^n eG = (bG)^n \in Q/G$. Therefore Q/G is a primary ideal.

$\square$

Proposition 3.11. If $Q^{'}$ is a primary hyperideal of R/G, then there is a primary ideal Q of R such that $Q^{'} = Q/G$, provided $1 \in G$.

Proof. Suppose that $Q^{'}$ is a primary hyperideal of R/G. So, we have an ideal Q of R such that $Q^{'} = Q/G$. To prove Q is a primary ideal, take $ab \in Q$ with $a \notin Q$. Thus, we have $abG \in Q/G$ with $aG \notin Q/G$. Otherwise, $aG \in Q/G$ implies $a = a1 \in Q$, a contradiction. Thus, by the definition of primary hyperideal, we have an $n \in \mathbb{N}$ such that $(bG)^n = b^n G \in Q/G$. Hence, $b^n = b^n 1 \in Q$.

$\square$

Proposition 3.12. If M is a maximal ideal of R, then M/G is a maximal hyperideal of R/G.

Proof. Suppose M is any maximal ideal of R. Consider the hyper ideal M/G. If $M/G \subsetneq J/G$, then there exists $jG \in J/G \setminus M/G$. Since $jG \notin M/G$, we have $j \notin M$. Also, $je \in J$ as $jG \in J/G$. Since M is maximal, we have $\langle M,\ j\rangle = R$. Also, since $M/G \subsetneq J/G$ $\langle M, je\rangle \subseteq J$. But, $\langle M,\ j\rangle /G = \langle M, je\rangle /G \subseteq J/G$. Thus, we have $J/G = R/G$.

Hence, M/G is a maximal hyperideal of R/G.

$\square$

Proposition 3.13. If $M^{'}$ is a maximal hyperideal of R/G, then there is a maximal ideal M of R such that $M^{'} = M/G$, provided $1 \in G$.

Proof. Clearly, we have an ideal M of R such that $M^{'} = M/G$. To prove M is maximal, let J be an ideal of R with $M \subsetneq J$. So, there exists $j \in J \setminus M$. Hence $jG \in J/G \setminus M/G$. If otherwise, $jG \in M/G$ then $j = j1 \in M$, a contradiction. Thus we have $J/G = R/G$. Therefore, $J = R$.

$\square$

Proposition 3.14. If R is any Noetherian ring, then R/G is a Noetherian hyperring, provided $1 \in G$.

Proof. Let $I_1^{'} \subseteq I_2^{'} \subseteq \ldots \subseteq I_n^{'} \subseteq \ldots$ be an ascending chain of hyperideals in R/G. For each hyperideal $I_k^{'}$, we have an ideal I_k of R such that $I_k^{'} = I_k/G$. Now take $x \in I_n$, we have $xG \in I_n/G \subseteq I_{n+1}/G$. Thus, $xG \in I_{n+1}/G$. As a result, there exists $i_{n+1} \in I_{n+1}$ such that $xG = i_{n+1}G$. So, we have $x = x1 = i_{n+1}g$ for some $g \in G$. And hence, $x \in I_{n+1}$. We have $I_n \subseteq I_{n+1}$ for each $n \in \mathbb{N}$. Consequently, $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$ is an ascending chain of ideals of R. Since R is a Noetherian ring , there is an $m \in \mathbb{N}$ such that $I_n = I_m$for each $n \geq m$. Thus, we have $I_n/G = I_m/G$ for each $n \geq m$. That is, $I_n^{'} = I_m^{'}$for each $n \geq m$. Hence R/G is Noetherian. $\square$

Proposition 3.15. If R/G is a Noetherian hyperring, then R is Noetherian ring, provided $1 \in G$.

Proof. Suppose that R/G is a Noetherian hyperring. Let $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ be an ascending chain of ideals in R. Then $I_1/G \subseteq I_2/G \subseteq \ldots \subseteq I_n/G \subseteq \ldots$ is an ascending chain of hyperideals of R/G. Since R/G is Noetherian, there exists $m \in \mathbb{N}$, such that $I_n/G = I_m/G$ for each $n \geq m$. Thus, we have $I_n = I_m$ for each $n \geq m$ as $1 \in G$. Hence R is Noetherian. $\square$

Proposition 3.16. If R is an Artinian ring, then R/G is an Artinian hyperring, provided $1 \in G$.

Proof. Similar to that of the Noetherian case. $\square$

Proposition 3.17. If R/G is an Artinian hyperring, then R is an Artinian ring, provided $1 \in G$.

Proof. Similar to that of the Noetherian case. $\square$

Theorem 3.1. *Let R be a commutative hyperring with 1, and G be a multiplicative subgroup of R. Then $R/G = \{rG | r \in R\}$ is a hyperring under the operations defined as follows:*

$$rG + sG = \{tG|\ t \in rg_1 + sg_2\ ;\ g_1, g_2 \in G\}$$

and

$$rG \cdot sG = rsG$$

.

Proof. To prove that the operations “ + ” and “ · ” are well defined, let $rG = r^{'}G$ and $sG = s^{'}G$. Take $tG \in rG+sG$ with $t \in rg_1+sg_2$ and having $g_1, g_2 \in G$. Since $rG = r^{'}G,\ sG = s^{'}G$ there exist $g_1^{'}, g_2^{'} \in G$ such

that $rg_1 = r^{'}g_1^{'},\ sg_2 = s^{'}g_2^{'}$. So, we have $rg_1 + sg_2 = r^{'}g_1^{'} + s^{'}g_2^{'}$. Since $t \in rg_1 + sg_2 = r^{'}g_1^{'} + s^{'}g_2^{'}$.
And, therefore $tG \in r^{'}G + s^{'}G$.
Thus, we have $rG + sG \subseteq r^{'}G + s^{'}G$. Similarly, we can prove $r^{'}G + s^{'}G \subseteq rG + sG$.
Thus, $rG + sG = r^{'}G + s^{'}G$.
Hence, we conclude that " $+$ " is well defined on R/G.
Now, we will prove that " $\cdot$ " is also well-defined.
Since $rG = r^{'}G$ and $sG = s^{'}G$ there exist $g_1, g_2 \in G$ such that $re = r^{'}g_1,\ se = s^{'}g_2$. So, $rse = r^{'}s^{'}(g_1g_2)$ with $g_1, g_2 \in G$. And therefore, $rsG = r^{'}s^{'}G$. In other words, $rG \cdot sG = r^{'}G \cdot s^{'}G$.
Hence, " $\cdot$ " is a binary operation on R/G.
To prove the associativity of " $+$ " , take aG, bG and cG in R/G. If $tG \in (aG + bG) + cG$, then $tG \in sG + cG$ for some $sG \in aG + bG$. Thus, we have $sG = xG$ with $x \in ag_1 + bg_2$ for some $g_1, g_2 \in G$, and $tG = yG$ with $y \in sg_1^{'} + cg_2^{'}$ for some $g_1^{'}, g_2^{'} \in G$. Since $sG = xG$ and $tG = yG$, we have $se = xg$ and $te = yg^{'}$, for some $g, g^{'} \in G$. So, we have $se \in ag_1g + bg_2g$. Therefore, by the distributive property, $sg_1^{'} \in ag_1gg_1^{'} + bg_2gg_1^{'}$. Also, $te \in sg_1^{'}g^{'} + cg_2^{'}g^{'}$. Consequently, we have $te \in \left(ag_1gg_1^{'}g^{'} + bg_2gg_1^{'}g^{'}\right) + cg_2^{'}g^{'}$. By the associativity of " $+$ " in R, we have $te \in ag_1gg_1^{'}g^{'} + \left(bg_2gg_1^{'}g^{'} + cg_2^{'}g^{'}\right)$.
And, hence, $te \in ag_3 + (bg_4 + cg_5)$, where $g_3 = g_1gg_1^{'}g^{'}, g_4 = g_2gg_1^{'}g^{'}, g_5 = g_2^{'}g^{'}$ are elements of G. Now, we can say $te \in ag_3 + u$, for some $u \in bg_4 + cg_5$. Thus, $tG \in aG + uG$ with $uG \in bG + cG$. Therefore, $tG \in aG + (bG + cG)$.
Hence, we have $(aG + bG) + cG \subseteq aG + (bG + cG)$. Similarly, we can prove $aG + (bG + cG) \subseteq (aG + bG) + cG$. That is, $(aG + bG) + cG = aG + (bG + cG)$.
Thus, associativity holds for " $+$ " in R/G.
Also, if $tG \in aG + bG$, then $tG = xG$, for some $xG \in aG + bG$, and therefore, there exist $g_1, g_2 \in G$ such that $x \in ag_1 + bg_2$.
Hence, we have $te = xg$, for some $g \in G$. This, in turn implies, $te \in (ag_1 + bg_2)\, g = ag_1g + bg_2g = bg_2g + ag_1g$. So, we have $tG \in bG + aG$. Thus, we have proved $aG + bG \subseteq bG + aG$. Similarly, we can prove $bG + aG \subseteq aG + bG$. Thus, $aG + bG = bG + aG$, which establishes the commutativity of " $+$ " in R/G.
As $0 \in R$, we have $0G \in R/G$ satisfying $aG + 0G = \{tG | t \in ag_1 + 0g_2, \text{for some } g_1, g_2 \in G\}$.
$= \{tG | t = ag_1, \text{ for some } g_1 \in G\} = \{aG\}$.
Hence, we have $aG + 0G = \{aG\} = 0G + aG$.

Now, take $aG \in R/G$, then, there exists $(-a)\, G \in R/G$ such that $0G \in aG + (-a)\, G$ as $0 \in ae + (-a)\, e$.
Let $aG \in bG + cG$. So, we have $aG = tG$, for some $tG \in bG + cG$. As a consequence, there exist $g_1, g_2 \in G$ satisfying $t \in bg_1 + cg_2$.
Hence, we have $ae \in bg_3 + cg_4$, for some $g_3, g_4 \in G$. As a result, $bg_3 \in ae + (-c)\, g_4$,
and hence, $bG = bg_3G \in aG + (-c)\, G$. Similarly, we can prove $cG \in aG + (-b)\, G$.

To establish the associativity of multiplication in R/G, we consider $aG \cdot (bG \cdot cG) = a\,(bc)\, G = (ab)\, cG = (aG \cdot bG) \cdot$ cG, as multiplication is associative in R.
Also, clearly, $aG \cdot 0G = a0G = 0G = 0G \cdot aG$, as $a0 = 0a = 0$
And, $aG \cdot bG = abG = baG = bG \cdot aG$, as $ab = ba$.
As $1 \in R$, $1G \in R/G$ such that $1G \cdot aG = 1aG = aG = aG \cdot 1G$.
To prove the distributivity, take $tG \in aG \cdot (bG + cG)$. So, we have $tG = aG \cdot xG$, for some $xG \in bG + cG$. Thus, there exists $g \in G$ such that $te = axg$. Also, $xe \in bg_1 + cg_2$, for some $g_1, g_2 \in G$. Thus, we have $te \in abg_1g + acg_2g$.
And, hence $tG = teG \in abG + acG = aG \cdot bG + aG \cdot cG$. Thus, we have $aG \cdot (bG + cG) \subseteq aG \cdot bG + aG \cdot cG$. If $sG \in aG \cdot bG + aG \cdot cG = abG + acG$ there exist $g_1, g_2 \in G$ and $x \in abg_1 + acg_2$ such that $sG = xG$. So, $se = xg$, for some $g \in G$. Which implies $se \in a\,(bg_1g + cg_2g)$, and therefore, we have $se = ay$, for some $y \in bg_1g + cg_2g$. As a result, we have $sG = aGyG$, and $yG \in bG + cG$. So, $sG \in aG \cdot (bG + cG)$. Thus, $aG \cdot bG + aG \cdot cG \subseteq aG \cdot (bG + cG)$ That is, $aG \cdot bG + aG \cdot cG = aG \cdot (bG + cG)$.
Hence distributivity holds.
So, R/G is a commutative hyperring with 1.

□

Theorem 3.2. *Let R be a commutative ring with 1, and G_1, G_2 be multiplicative subgroups of R with $G_1 \subseteq G_2$. Then,*

1. *G_2/G_1 is a multiplicative subgroup of R/G_1,*
2. *$R/G_2 \cong (R/G_1)\,/(G_2/G_1)$.*

Proof.

1. Since R is commutative, G_1 is a normal subgroup of G_2, and $G_2/G_1 \subseteq R/G_1$. Now, as quotient structure G_2/G_1 is a group under multiplication, and hence, G_2/G_1 is a multiplicative subgroup of R/G_1.
2. By the previous theorem, $(R/G_1)\,/(G_2/G_1)$ is a hyperring. To prove the above, Define $\Phi : (R/G_1)\,/(G_2/G_1) \longrightarrow R/G_2$ by $\Phi\,(rG_1 \cdot (G_2/G_1)) = rG_2$
   To show the well definedness of Φ, take $rG_1\,(G_2/G_1), sG_1\,(G_2/G_1)$ in $(R/G_1)\,/(G_2/G_1)$ such that $rG_1\,(G_2/G_1) = sG_1\,(G_2/G_1)$. So, we have $rG_1eG_1 = sG_1g_2G_1$, for some $g_2 \in G_2$.
   Hence, $reG_1 = sg_2G_1$, for some $g_2 \in G_2$. Consequently, there exists $g_1 \in G_1$ such that $re = sg_2g_1$. Since $g_2, g_1 \in G_2$, we have $rG_2 = sG_2$. Thus, $\Phi\,(rG_1\,(G_2/G_1)) = \Phi\,(sG_1\,(G_2/G_1))$. That is, Φ is well defined.
   To prove the injectivity of Φ, take rG_2 and sG_2 in R/G_2 such that $rG_2 = sG_2$, then $re = sg_2$, for some $g_2 \in G_2$. Thus, we have $reG_1 = sg_2G_1$, which, in turn, implies

$rG_1eG_1 = sG_1g_2G_1$. From this, we can conclude that $rG_1(G_2/G_1) = sG_1(G_2/G_1)$, and, hence Φ is one to one.
To prove the surjectivity of Φ, pick $rG_2 \in R/G_2$ then, there exists $(rG_1)G_2/G_1 \in (R/G_1)/(G_2/G_1)$ such that $\Phi((rG_1)G_2/G_1) = rG_2$. So, Φ is onto.
To show that Φ is a homomorphism, take $aG_1(G_2/G_1), bG_1(G_2/G_1)$ in $(R/G_1)/(G_2/G_1)$.
Consider $\Phi(aG_1(G_2/G_1) \cdot bG_1(G_2/G_1)) = \Phi(aG_1 \cdot bG_1(G_2/G_1)) = \Phi(abG_1(G_2/G_1)) = abG_2 = aG_2 \cdot bG_2 = \Phi(aG_1(G_2/G_1)) \cdot \Phi(bG_1(G_2/G_1))$.
Now, take
$\Phi(tG_1(G_2/G_1)) \in \Phi(aG_1(G_2/G_1) + bG_1(G_2/G_1))$ with $tG_1(G_2/G_1) \in aG_1(G_2/G_1) + bG_1(G_2/G_1)$. Then, we have $tG_1(G_2/G_1) = xG_1(G_2/G_1)$ with $xG_1 \in aG_1g_2G_1 + bG_1g_2'G_1$, for some $g_2, g_2' \in G_2$. So, $xG_1 \in ag_2G_1 + bg_2'G_1$. Thus, we have $xG_1 = yG_1$ with $y = ag_2g_1 + bg_2g_1'$ for some $g_1, g_1' \in G_1$.
Consequently, $xe = yg_1''$ for some $g_1'' \in G_1$, and, as a result of which, we have $xe = ag_2g_1g_1'' + bg_2g_1'g_1''$. Thus, $xeG_2 = xG_2 \in aG_2 + bG_2$. But, $\Phi(tG_1(G_2/G_1)) = \Phi(xG_1(G_2/G_1)) = xG_2 \in aG_2 + bG_2 = \Phi(aG_1(G_2/G_1)) + \Phi(bG_1(G_2/G_1))$.
Thus, $\Phi(aG_1(G_2/G_1) + bG_1(G_2/G_1)) \subseteq \Phi(aG_1(G_2/G_1)) + \Phi(bG_1(G_2/G_1))$.
Also, we have, $\Phi(aG_1(G_2/G_1)) = aG_2$ and $\Phi(bG_1(G_2/G_1)) = bG_2$. Now, take $rG_2 \in aG_2 + bG_2$, $r = ag_2 + bg_2'$, for some $g_2, g_2' \in G_2$. As a result, we have $rG_1 = \left(ag_2 + bg_2'\right)G_1 \subseteq ag_2G_1 + bg_2'G_1 = aG_1g_2G_1 + bG_1g_2'G_1$. And, hence $rG_1(G_2/G_1) \in aG_1(G_2/G_1) + bG_1(G_2/G_1)$. Thus, we have $\Phi(rG_1(G_2/G_1)) = rG_2 \in \Phi(aG_1(G_2/G_1) + bG_1(G_2/G_1))$. Therefore, $aG_2 + bG_2 = \Phi(aG_1(G_2/G_1)) + \Phi(bG_1(G_2/G_1)) \subseteq \Phi(aG_1(G_2/G_1) + bG_1(G_2/G_1))$
Thus, $\Phi(aG_1(G_2/G_1) + bG_1(G_2/G_1)) = \Phi(aG_1(G_2/G_1)) + \Phi(bG_1(G_2/G_1))$.
Also, $\Phi(1G_1(G_2/G_1)) = 1G_2$.
Hence, Φ is an isomorphism.
i.e. $(R/G_1)/(G_2/G_1) \cong R/G_2$.

□

Theorem 3.3. *Let f be a homomorphism from R onto S, and G be a multiplicative subgroup of R. Then,*

1. *$f(G)$ is a multiplicative subgroup of S,*
2. *$(R/G)/(\text{ker}f/G) \cong S/f(G)$.*

Proof.

1. Clearly, the multiplication given to S is a binary operation on $f(G)$, and also, associativity of multiplication for $f(G)$ is inherited from S. We have "$f(e)$" for the identity of $f(G)$, where "e" is the identity of G, as $f(e)f(g) = f(eg) = f(g) = f(ge) = f(g)f(e)$, for each, $f(g) \in f(G)$. Also, if $f(g) \in f(G)$, then $f(g)f(g^{-1}) = f(gg^{-1}) = f(e) = f(g^{-1}g) = f(g^{-1})f(g)$, and hence, $f(g)^{-1} = f\left(g^{-1}\right)$. Thus, $f(G)$ is a multiplicative subgroup of S.

2. Now, define $\Phi : R/G \longrightarrow S/f(G)$ by $\Phi(aG) = f(a)f(G)$. To prove the well-definedness of Φ, take $aG = bG$, then $ae = bg$, for some $g \in G$. Since f is a homomorphism, we have $f(a)f(e) = f(b)f(g)$, with $f(g) \in f(G)$. Thus, $f(a)f(G) = f(b)f(G)$. And, therefore, Φ is well defined.
To establish the surjectivity of Φ, take $xf(G) \in S/f(G)$ with $x \in S$, then, there exist $y \in R$ such that $y = f(x)$ as f is onto. So, $\Phi(xG) = f(x)f(G) = yf(G)$. Thus, Φ is an onto function.
Now, to prove Φ is a homomorphism, take aG, bG and tG in R/G such that $\Phi(tG) \in \Phi(aG + bG)$ with $tG \in aG + bG$. So, there exist $g_1, g_2 \in G$ such that $t = ag_1 + bg_2$. Since f is a homomorphism, we have $f(t) = f(ag_1) + f(bg_2) = f(a)f(g_1) + f(b)f(g_2)$. Now, consider $\Phi(tG) = f(t)f(G) \in f(a)f(G) + f(b)f(G) = \Phi(aG) + \Phi(bG)$. Thus, we have $\Phi(aG + bG) \subseteq \Phi(aG) + \Phi(bG)$.
Now, take $xf(G)$ in $S/f(G)$ such that $xf(G) \in \Phi(aG) + \Phi(bG) = f(a)f(G) + f(b)f(G)$. Consequently, there exist $g_1, g_2 \in G$ such that $x = f(a)f(g_1) + f(b)f(g_2)$. Thus, we have $x = f(ag_1 + bg_2)$. $\Rightarrow xf(G) = f(ag_1 + bg_2)f(G) = \Phi((ag_1 + bg_2)G) \in \Phi(aG + bG)$.
Hence, $\Phi(aG) + \Phi(bG) \subseteq \Phi(aG + bG)$.
Thus, $\Phi(aG + bG) = \Phi(aG) + \Phi(bG)$.
Now, let $aG, bG \in R/G$, and consider $\Phi(aG \cdot bG) = \Phi(abG) = f(ab)f(G) = f(a)f(b)f(G) = f(a)f(G)f(b)f(G) = \Phi(aG)\Phi(bG)$.
Hence, Φ is a homomorphism.
Also, $\Phi(1G) = f(1)f(G) = 1'f(G)$, where $1, 1'$ are multiplicative identities of R and S, respectively. Thus, Φ is an onto homomorphism.
By the 1st isomorphism theorem for the hyperrings (Velrajan and Asokkumar, 2010), $(R/G)/\ker\Phi \cong S/f(G)$.
Here, $\ker\Phi = \{rG \in R/G : \Phi(rG) = 0f(G)\} = \{rG \in R/G : f(r)f(G) = 0\} = \{rG \in R/G : f(r)f(e) = f(\text{re}) = 0\} = \{reG \in R/G : re \in \ker f\} = \ker f/G$.
Hence, $(R/G)/(\ker f/G) \cong S/f(G)$.

□

Corollary 3.1. *Let I an ideal of R, and G be a multiplicative subgroup of R. Then,*

1. $G/I = \{g + I | g \in G\}$ *is a multiplicative subgroup of* R/I,
2. $(R/G)\,/\,(I/G) \cong (R/I)\,/\,(G/I)$.

Proof. We apply the previous theorem with $f = nat_I : R \longrightarrow R/I$. Firstly, $nat_I\,(G) = G/I$ is a multiplicative subgroup of R/I, and $\ker(nat_I) = I$. So, by the previous theorem, $(R/G)\,/\,(\ker(nat_I)\,/G) \cong (R/I)\,/\,(G/I)$.
i.e. $(R/G)\,/\,(I/G) \cong (R/I)\,/\,(G/I)$
.

□

4. Conclusion

Firstly, we identified certain necessary and sufficient conditions for R/G to be a hyperdomain and hyperfield, and then proved that a direct product of Krasner's induced quotient hyperrings is a Krasner's induced quotient hyperring. Then we established a relationship between the ideals of R and the hyperideals of R/G. Consequently, we came up with relationships between the prime, primary, maximal ideals of R and prime, primary, maximal hyperideals of R/G, respectively. We found a method of constructing a new hyperring out of a given hyperring, and using this; we produced two theorems characterizing the isomorphicity of certain distinct quotient constructions. We hope that these theorems will play an important role, in the future, in the classification of finite hyperrings.

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