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Research Article

A new stochastic restricted two-parameter estimator in multiple linear regression model

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Abstract: In this paper, we proposed a biased estimator, a new stochastic restricted two-parameter estimator (NSRTPE), for the multiple linear regression model to tackle the multicollinearity problem when the stochastic restrictions are available. Necessary and sufficient conditions for the superiority of the proposed estimator over the ordinary least square estimator (OLSE), ridge estimator (RE), Liu estimator (LE), almost unbiased Liu estimator (AULE), modified new two-parameter estimator (MNTPE), mixed estimator (ME), stochastic restricted Liu estimator (SRLE) were derived in the mean square error matrix (MSEM) criterion. Finally, we showed the superiority of the estimator proposed using a simulation study and a real-world example in the scalar mean square error (SMSE) criterion.

Keywords: Multiple linear regression, Multicollinearity, Stochastic restriction, Two-parameter estimator, Mean square error matrix

1. Introduction

We consider the multiple linear regression model

$$y = X\beta + \epsilon, \quad \epsilon \sim N\left(0, \sigma^2 I\right) \tag{1.1}$$

where y is an $n \times 1$ observable random vector, X is a $n \times p$ known design matrix of rank p , β is a $p \times 1$ vector of unknown parameters, and ϵ is a $n \times 1$ vector of disturbances. The ordinary least square estimator (OLSE) for model is given by (1.1)

$$\widehat{\beta}_{OLSE} = S^{-1}X^{'}y, \tag{1.2}$$

where $S = X^{'}X$. In the case of multicollinearity, the Ordinary Least Squares method produces estimates with significant variances, wide confidence intervals, unreliable tests, and incorrect signs. Several researchers have proposed alternative estimators instead of the Ordinary Least Square Estimator (OLSE) to confront the multicollinearity problem. In order to solve the multicollinearity problem, Hoerl and Kennard (1970) firstly proposed the Ridge Estimator (RE). Followed by Hoerl and Kennard (1970), the Liu Estimator (LE) by Liu (1993), and the Almost Unbiased Liu Estimator (AULE) by Akdeniz and Kaçiranlar (1995) have been proposed to solve multicollinearity. Recently, Ahmad and Aslam (2020) proposed Modified New Two Parameter Estimator (MNTPE).

The RE proposed by Hoerl and Kennard (1970) is defined as

$$\widehat{\beta}_{RE}\left(k\right) = W_k \widehat{\beta}_{OLSE}, \tag{1.3}$$

where $W_k = \left(I + kS^{-1}\right)^{-1}$.

The bias vector and mean square error (MSEM) matrix of RE can be obtained as

$$B\left(\widehat{\beta}_{RE}\left(k\right)\right) = -k\left(S + kI\right)^{-1} \text{ and} \tag{1.4}$$

$$MSEM\left(\widehat{\beta}_{RE}\left(k\right)\right) = \sigma^2 W_k S^{-1} W_k + k^2\left(S + kI\right)^{-1}\beta\beta'\left(S + kI\right)^{-1}, \tag{1.5}$$

respectively.

The LE proposed by Liu (1993) is defined as

$$\hat{\beta}_{LE}(d) = F_d \hat{\beta}_{OLSE}, \tag{1.6}$$

where $F_d = \left(S + I\right)^{-1}\left(S + dI\right)$.

The bias vector and MSEM of LE can be obtained as

$$B\left(\hat{\beta}_{LE}(d)\right) = (d-1)\left(S + I\right)^{-1}\beta \text{ and} \tag{1.7}$$

$$MSEM\left(\hat{\beta}_{LE}(d)\right) = \sigma^2 F_d S^{-1} F_d^{'} + (1-d)^2\left(S + I\right)^{-1}\beta\beta^{'}\left(S + I\right)^{-1}, \tag{1.8}$$

respectively.

The AULE proposed by Akdeniz and Kaçiranlar (1995) is defined as

$$\hat{\beta}_{AULE}(d) = T_d \hat{\beta}_{OLSE}, \tag{1.9}$$

where $T_d = \left[I - (1-d)^2\left(S + I\right)^{-2}\right]$

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The bias vector and MSEM of AULE can be obtained as

$$B\left(\hat{\beta}_{AULE}(d)\right) = -\left(1-d\right)^2\left(S+I\right)^{-2}\beta \text{ and} \quad (1.10)$$

$$MSEM\left(\hat{\beta}_{AULE}(d)\right) = \sigma^2 T_d S^{-1} T_d^{'} + \left(1-d\right)^4\left(S+I\right)^{-2}\beta\beta^{'}\left(S+I\right)^{-2}, \quad (1.11)$$

respectively.

The MNTPE proposed by Ahmad and Aslam (2020) is defined as

$$\beta_{MNTPE}^{'}\left(k,d\right) = L_{k,d}\hat{\beta}_{OLSE}, \quad (1.12)$$

where $L_{k,d} = \left(S+I\right)^{-1}\left(S+dI\right)\left(S+kdI\right)^{-1}S$

The bias vector and MSEM of MNTPE can be obtained as

$$B\left(\hat{\beta}_{MNTPE}(k,d)\right) = \left(L_{k,d}-I\right)\beta \text{ and} \quad (1.13)$$

$$MSEM\left(\hat{\beta}_{MNTPE}(k,d)\right) = \sigma^2 L_{k,d}S^{-1}L_{k,d}^{'} + \left(L_{k,d}-I\right)\beta\beta^{'}\left(L_{k,d}-I\right)^{'}, \quad (1.14)$$

respectively.

Another method to deal with multicollinearity problems is to consider parameter estimation with some restrictions on the unknown parameters, which may be exact or stochastic restrictions (Rao and Touterburg, 1995). In addition to the sample model (1.1), let us be given some prior information about βin the form of a set of q independent stochastic linear restrictions as follows.

$$h = H\beta + \upsilon, \upsilon \sim N\left(0, \sigma^2\Omega\right), \quad (1.15)$$

where h is a known stochastic vector, H is an $q\times p$ of full row rank $q \leq p$ with known element, υ is a $q\times 1$ random vector of disturbances, and Ω is assumed to be known and positive definite. Moreover, it is assumed that υ is stochastically independent of ϵ , i.e.,$E\left(\epsilon\upsilon^{'}\right) = 0$. Then, by combining the sample model (1.1) and the stochastic restriction (1.15), the Mixed Estimator (ME) is proposed by Theil and Goldberger (1961) as follows.

$$\hat{\beta}_{ME} = \hat{\beta}_{OLSE} + S^{-1}H^{'}(HS^{-1}H^{'}+\Omega)^{-1}(h - H\hat{\beta}_{OLSE}) \quad (1.16)$$

The MSEM of ME can be obtained as

$$MSEM\left(\hat{\beta}_{ME}\right) = \sigma^2\left(S^{-1}-G\right) = \sigma^2 A, \quad (1.17)$$

where $G = S^{-1}H^{'}\left(\Omega + HS^{-1}H^{'}\right)HS^{-1}$ and $A = S^{-1} - G = \left(S + H\Omega^{-1}H^{'}\right)^{-1}$.

By replacing ME in the place of OLSE in LE, Hubert and Wijekoon (2006) proposed the stochastic restricted Liu estimator (SRLE) as follows.

$$\hat{\beta}_{SRLE}(d) = F_d\hat{\beta}_{ME} \quad (1.18)$$

The bias vector and MSEM of SRLE are given by

$$B\left(\hat{\beta}_{SRLE}(d)\right) = \left(F_d - I\right)\beta \text{ and} \quad (1.19)$$

$$MSEM\left(\beta_{SRLE}(d)\right) = \sigma^2 F_d S^{-1}F_d^{'} - \sigma^2 F_d G F_d^{'} + \left(F_d - I\right)\beta\beta^{'}\left(F_d - I\right)^{'}, \quad (1.20)$$

respectively.

The hope is that the combination of two different estimators might inherit the advantages of both estimators. Therefore, in this research, we propose a new estimator by combining ME and MNTPE.

2. The Proposed Estimator and Its Stochastic Properties

Following Hubert and Wijekoon (2006), we propose a new estimator named as New Stochastic Restricted Two Parameter Estimator (NSRTPE) by replacing ME in places of OLSE in the Modified New Two Parameter Estimator (MNTPE) as follows.

$$\hat{\beta}_{NSRTPE}(k,d) = L_{k,d}\hat{\beta}_{ME} \quad (2.1)$$

The bias vector, dispersion matrix, and MSEM of $\hat{\beta}_{NSRTPE}(k,d) = L_{k,d}\hat{\beta}_{ME}$ can be obtained as

$$B\left(\hat{\beta}_{NSRTPE}(k,d)\right) = \left(L_{k,d}-I\right)\beta \quad (2.2)$$

$$D\left(\hat{\beta}_{NSRTPE}(k,d)\right) = \sigma^2 L_{k,d}S^{-1}L_{k,d}^{'} - \sigma^2 L_{k,d}GL_{k,d}^{'} \quad (2.3)$$

and

$$MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) = \sigma^2 L_{k,d}S^{-1}L_{k,d}^{'} - \sigma^2 L_{k,d}GL_{k,d}^{'} + \left(L_{k,d}-I\right)\beta\beta^{'}\left(L_{k,d}-I\right)^{'}, \quad (2.4)$$

respectively.

3. Mean Square Error Matrix Comparison

This section compares the performance of the proposed estimator with OLSE, ME, RE, LE, AULE, SRLE, and MNTPE.

3.1. MSEM comparison between RE and NSRTPE

In this section, the RE and NSRTPE will be compared. The MSEM difference is

$$MSEM\left(\hat{\beta}_{RE}(k)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) = \sigma^2 W_k S^{-1} W_k - \sigma^2 L_{k,d} S^{-1} L_{k,d}^{'} + \sigma^2 L_{k,d} G L_{k,d}^{'} + B\left(\hat{\beta}_{RE}(k)\right) B^{'}\left(\hat{\beta}_{RE}(k)\right) - B\left(\hat{\beta}_{NSRTPE}(k,d)\right) B^{'}\left(\hat{\beta}_{NSRTPE}(k,d)\right) \quad (3.1)$$

Now the following theorem can be stated.

Theorem 3.1:
When $\lambda_{max}\left[\left(L_{k,d}S^{-1}L_{k,d}^{'}\right)\left(W_k S^{-1} W_k\right)^{-1}\right] < 1$, the estimator NSRTPE is superior to RE in the mean squared error matrix sense if and only if

$$\beta^{'}\left(L_{k,d}-I\right)^{'}\left(\sigma^2 D_1 + \left(W_k - I\right)\beta\beta^{'}\left(W_k - I\right)^{-1}\left(L_{k,d}-I\right)\beta\right) \le 1$$

where $D_1 = W_k S^{-1} W_k - L_{k,d}S^{-1}L_{k,d}^{'} + L_{k,d}GL_{k,d}^{'}$ and $\lambda_{max}\left[\left(L_{k,d}S^{-1}L_{k,d}^{'}\right)\left(W_k S^{-1} W_k\right)^{-1}\right]$ is the largest eigenvalue of the matrix $\left(L_{k,d}S^{-1}L_{k,d}^{'}\right)\left(W_k S^{-1} W_k\right)^{-1}$.

Proof: One can clearly say that the matrix $L_{k,d}GL_{k,d}^{'}$ is positive definite. According to lemma 1 (Appendix), it can be said that the matrix D_1 is positive definite if and only if $\lambda_{max}\left[\left(L_{k,d}S^{-1}L_{k,d}^{'}\right)\left(W_k S^{-1} W_k\right)^{-1}\right] < 1$.
Now, based on Lemma 2 (Appendix), we can say that $MSEM\left(\hat{\beta}_{RE}(k)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) \ge 0$ if and only if
$\beta^{'}\left(L_{k,d}-I\right)^{'}(\sigma^2 D_1 + \left(W_k - I\right)\beta\beta^{'}\left(W_k - I\right))^{-1}(L_{k,d} - I)\beta \le 1$.
This completes the proof.

3.2. MSEM comparison between LE and NSRTPE

In order to compare the LE and NSRTPE in terms of the MSEM matrix, we investigate the following difference.

$$MSEM\left(\hat{\beta}_{LE}(d)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) = \sigma^2 F_d S^{-1} F_d^{'} - \sigma^2 L_{k,d} S^{-1} L_{k,d}^{'} + \sigma^2 L_{k,d} G L_{k,d}^{'} + B\left(\hat{\beta}_{LE}(d)\right) B\left(\hat{\beta}_{LE}(d)\right)^{'} - B\left(\hat{\beta}_{NSRTPE}(k,d)\right) B\left(\hat{\beta}_{NSRTPE}(k,d)\right)^{'} \quad (3.2)$$

Now, one can state the following theorem.

Theorem 3.2 The NSRTPE is superior to LE in the mean squared error matrix sense if and only if

$$B^{'}\left(\hat{\beta}_{NSRTPE}(k,d)\right)\left(\sigma^2 D_2 + B\left(\hat{\beta}_{LE}(d)\hat{\beta}_{LE}(d)\right)^{'}\right)^{-1} B\left(\hat{\beta}_{NSRTPE}(k,d)\right) \le 1$$

where $D_2 = F_d S^{-1} F_d^{'} - L_{k,d}S^{-1}L_{k,d}^{'} + L_{k,d}GL_{k,d}^{'}$.

Proof:Let us consider

$$F_d S^{-1} F_d^{'} - L_{k,d}S^{-1}L_{k,d}^{'}$$

$$\begin{aligned}
&= F_d S^{-1} F_d^{'} - (S+I)^{-1}(S+dI)(S + kdI)^{-1} S S^{-1} S\,(S+kdI)^{-1}(S+dI)(S+I)^{-1} \\
&= F_d (S+kdI)^{-1}\left((S+kdI)S^{-1}(S+kdI) - S\right)(S + kdI)^{-1} F_d^{'} \\
&= F_d (S+kdI)^{-1}\left(S + kdI + kdI + k^2d^2S^{-1} - S\right)(S + kdI)^{-1} F_d^{'} \\
&= kdF_d (S+kdI)^{-1}\left(I + kdS^{-1}\right)(S+kdI)^{-1} F_d^{'}
\end{aligned}$$

Since $0 < d < 1$ and $k \ge 0$, the matrix $F_d S^{-1} F_d^{'} - L_{k,d}S^{-1}L_{k,d}^{'}$ is positive definite. Therefore, the matrix $D_2 = F_d S^{-1} F_d^{'} - L_{k,d}S^{-1}L_{k,d}^{'} + L_{k,d}GL_{k,d}^{'}$ is positive definite since $L_{k,d}^{'}GL_{k,d}^{'} > 0$. Now, based on Lemma 2, it can be said that $MSEM\left(\hat{\beta}_{LE}(d)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) \ge 0$ if and only if $B^{'}(\hat{\beta}_{NSRTPE}(k,d))(\sigma^2 D_2 + B(\hat{\beta}_{LE}(d)\hat{\beta}_{LE}(d))^{'})^{-1} B(\hat{\beta}_{NSRTPE}(k,d)) \le 1$
This completes the proof.

3.3. MSEM comparison between AULE and NSRTPE

In this section, the AULE and NSRTPE will be compared. The MSEM difference is

$$MSEM\left(\hat{\beta}_{AULE}(d)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) = \sigma^2 T_d S^{-1} T_d - \sigma^2 L_{k,d} S^{-1} L_{k,d}^{'} + \sigma^2 L_{k,d} G L_{k,d}^{'} + B\left(\hat{\beta}_{AULE}(d)\right) B\left(\hat{\beta}_{AULE}(d)\right)^{'} - B\left(\hat{\beta}_{NSRTPE}(k,d)\right) B\left(\hat{\beta}_{NSRTPE}(k,d)\right)^{'} \quad (3.3)$$

Now the following theorem can be stated.

Theorem 3.3:
When $\lambda_{max}\left[\left(L_{k,d}S^{-1}L_{k,d}^{'}\right)\left(T_d S^{-1} T_d\right)^{-1}\right] < 1$ the estimator NSRTPE is superior to AULE in the mean squared error matrix sense if and only if $\beta^{'}(L_{k,d} - I)^{'}(\sigma^2 D_3 + (T_d - I)\beta\beta^{'}(T_d - I))^{-1}(L_{k,d} - I)\beta \le 1$,

where $D_3 = T_d S^{-1} T_d - L_{k,d} S^{-1} L'_{k,d} + L_{k,d} G L'_{k,d}$ and $\lambda_{max}\left[\left(L_{k,d} S^{-1} L'_{k,d}\right)\left(L_{k,d} S^{-1} L'_{k,d}\right)^{-1}\right]$ is the largest eigenvalue of the matrix $\left(L_{k,d} S^{-1} L'_{k,d}\right)\left(T_d S^{-1} T_d\right)^{-1}$.

Proof: One can say that the matrix $L_{k,d} G L'_{k,d}$ is a positive definite matrix. According to lemma 1, it can be said that the matrix D_3 is a positive definite matrix if and only if $\lambda_{max}\left[\left(L_{k,d} S^{-1} L'_{k,d}\right)\left(T_d S^{-1} T_d\right)^{-1}\right] < 1$. Now, based on the Lemma 3, we can say that $MSEM\left(\hat{\beta}_{AULE}(d)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) \geq 0$ if and only if $\beta'(L_{k,d} - I)'(\sigma^2 D_3 + (T_d - I)\beta\beta'(T_d - I)^{-1})^{-1}(L_{k,d} - I)\beta \leq 1$
This completes the proof.

3.4. MSEM comparison between SRLE and NSRTPE

We consider the MSEM difference between SRLE and NSRTPE as:

$$\begin{aligned}
&MSEM\left(\hat{\beta}_{SRLE}(d)\right) - MSEM\left(\hat{\beta}_{NSRTPE}(k,d)\right) \\
&= \sigma^2 F_d S^{-1} F'_d - \sigma^2 F_d G F'_d - \sigma^2 L_{k,d} S^{-1} L'_{k,d} \\
&+ \sigma^2 L_{k,d} G L'_{k,d} + B\left(\hat{\beta}_{SRLE}(d)\right) B'\left(\hat{\beta}_{SRLE}(d)\right) \\
&- B\left(\hat{\beta}_{NSRTPE}(k,d)\right) B'\left(\hat{\beta}_{NSRTPE}(k,d)\right)
\end{aligned} \tag{3.4}$$

Now, the following theorem can be stated.

Theorem 3.4: The NSRTPE is superior to SRLE in the mean squared error matrix sense if and only if $B'(\hat{\beta}_{NSRTPE}(k,d))(\sigma^2 D_4 + B(\hat{\beta}_{SRLE}(d)\hat{\beta}_{SRLE}(d))')^{-1} B(\hat{\beta}_{NSRTPE}(k,d)) \leq 1$, where $D_4 = F_d S^{-1} F'_d - F_d G F'_d - F_{k,d} S^{-1} F'_{k,d} + F_d G F'_{k,d}$.
Proof:Let us consider

$$\begin{aligned}
&F_d S^{-1} F'_d - F_d G F'_d - F_{k,d} S^{-1} L'_{k,d} + F_{k,d} G L'_{k,d} \\
&= F_d\left(S^{-1} - G\right) F'_d - L_{k,d}\left(S^{-1} - G\right) L'_{k,d} \\
&= F_d A F'_d - L_{k,d} A L'_{k,d} \\
&= F_d A F'_d - F_d\left(I + kdS^{-1}\right)^{-1} A\left(I + kdS^{-1}\right)^{-1} F'_d \\
&= F_d\left(I + kdS^{-1}\right)^{-1}\left[\left(I + kdS^{-1}\right) A\left(I + kdS^{-1}\right) - A\right]\left(I + kdS^{-1}\right)^{-1} F'_d \\
&= F_d\left(I + kdS^{-1}\right)^{-1}\left[A + kdAS^{-1} + kdS^{-1}A + k^2 d^2 S^{-2} - A\right]\left(I + kdS^{-1}\right)^{-1} F'_d \\
&= kdF_d\left(I + kdS^{-1}\right)^{-1}\left[AS^{-1} + S^{-1}A + kdS^{-2}\right](I + kdS^{-1})^{-1} F'_d
\end{aligned}$$

It can be said that the matrix $F_d S^{-1} F'_d - F_d G F'_d - F_{k,d} S^{-1} L'_{k,d} + F_{k,d} G L'_{k,d}$ is positive definite since $F_d > 0, \left(I + kdS^{-1}\right)^{-1} > 0$ and $\left[AS^{-1} + S^{-1}A + kdS^{-2}\right] > 0$. Now, according to lemma 2, one can say that one can say that $MSEM\left(\hat{\beta}_{SRLE}(d)\right) - MSE\left(\hat{\beta}_{NRSTPE}(k,d)\right) \geq 0$ if and only if $B'(\hat{\beta}_{NSRTPE}(k,d))(\sigma^2 D_4 + B(\hat{\beta}_{SRLE}(d))B(\hat{\beta}_{SRLE}(d))')^{-1} B(\hat{\beta}_{NSRTPE}(k,d)) \leq 1$.
This completes the proof.

3.5. MSEM M comparison between MNTPE and NSRTPE

We consider the MSEM difference between MNTPE and NSRTPE as:

$$\begin{aligned}
&MSE\left(\hat{\beta}_{MNTPE}(k,d)\right) - MSE\left(\hat{\beta}_{NSRTPE}(k,d)\right) \\
&= \sigma^2 L_{k,d} G L'_{k,d}
\end{aligned} \tag{3.5}$$

Since the matrix $\sigma^2 L_{k,d} G L'_{k,d}$ is nonnegative definite, it can be said that $MSEM\left(\hat{\beta}_{MNTPE}(k,d)\right) - MSE\left(\hat{\beta}_{NSRTPE}(k,d)\right) \geq 0$, which means that the estimator NSRTPE is always superior to MNTPE in the mean squared error matrix sense.

3.6. MSEM comparison between OLSE and NSRTPE

In this subsection, the estimator NSRTPE will be compared with OLSE. We consider the MSEM difference between $\hat{\beta}_{OLSE}$ and $\hat{\beta}_{NSRTPE}(k,d)$ as

$$\begin{aligned}
&MSE\left(\hat{\beta}_{OLSE}\right) - MSE\left(\hat{\beta}_{NSRTPE}(k,d)\right) \\
&= \sigma^2 S^{-1} - \sigma^2 L_{k,d} S^{-1} L'_{k,d} + \sigma^2 L_{k,d} G L'_{k,d} \\
&+ B\left(\hat{\beta}_{NSRTPE}(k,d)\right) B'\left(\hat{\beta}_{NSRTPE}(k,d)\right)
\end{aligned} \tag{3.6}$$

Now, the following theorem can be stated.

Theorem 3.5 The OLSE is superior to NSRTPE in the mean square error matrix sense if and only if $B'\left(\hat{\beta}_{NSRTPE}(k,d)\right) D_5^{-1} B\left(\hat{\beta}_{NSRTPE}(k,d)\right) \leq \sigma^2$

where $D_5 = \sigma^2 S^{-1} - \sigma^2 L_{k,d} S^{-1} L'_{k,d} + \sigma^2 L_{k,d} G L'_{k,d}$.

Proof To prove this theorem, first, we need to show that the matrix D_5 is a positive definite. Since the matrix S is positive definite, there exists an orthogonal matrix P and a positive definite diagonal matrix $\Lambda = diag\left(\lambda_1, \lambda_2, \ldots\ldots, \lambda_p\right)$ such that $P'SP = \Lambda$, with $p'p = pp' = I$, where $\lambda_1 \geq \lambda_2 \geq \ldots\ldots \lambda_p > 0$ are the ordered eigenvalues of S.

Now, we consider

$$tr(S^{-1}-L_{k,d}S^{-1}L_{k,d}^{'}) = \sum_{i=1}^{p}\left[\frac{1}{\lambda_i} - \frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i}\right]$$

Now, in order to prove that $S^{-1} - L_{k,d}S^{-1}L_{k,d}^{'} > 0$, we need to show that $\frac{1}{\lambda_i} - \frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i} > 0$.

Consider

$$\begin{aligned}
&\frac{1}{\lambda_i} - \frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i}\\
=&\frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i}[(\lambda_i+1)^2(\lambda_i+kd)^2 -\\
&(\lambda_i+d)^2\lambda_i^2]\\
=&\frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i}[(\lambda_i+1)^2(\lambda_i+kd)^2 -\\
&(\lambda_i+d)^2\lambda_i^2]\\
=&\frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i}[(\lambda_i+1)(\lambda_i+kd) +\\
&(\lambda_i+d)\lambda_i][(\lambda_i+1)(\lambda_i+kd) - (\lambda_i+d)\lambda_i]\\
=&\frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i}[(\lambda_i+1)(\lambda_i+kd) +\\
&(\lambda_i+d)\lambda_i][kd(\lambda_i+1)+(1-d)\lambda_i]
\end{aligned} \tag{3.7}$$

It is clear that $\frac{1}{\lambda_i} - \frac{(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2\lambda_i} > 0$ since $0 < d < 1$ and $k > 0$.

Therefore, the matrix $S^{-1} - L_{k,d}S^{-1}L_{k,d}^{'} > 0$ so that $D_5 > 0$ since $L_{k,d}GL_{k,d}^{'} > 0$. Now, according to Lemma 3, it can be said that the OLSE is superior to NSRTPE in the mean square error matrix sense if and only if $B^{'}\left(\hat{\beta}_{NSRTPE}(k,d)\right)D_5^{-1}B\left(\hat{\beta}_{NSRTPE}(k,d)\right) \le \sigma^2$.

This completes the proof.

3.7. MSEM comparison between ME and NSRTPE

We consider the MSEM difference between ME and NSRTPE as:

$$\begin{aligned}
&MSE\left(\hat{\beta}_{ME}\right) - MSE\left(\hat{\beta}_{NSRTPE}(k,d)\right)\\
&= \sigma^2 A - \sigma^2 L_{k,d}AL_{k,d}^{'}\\
&+ B\left(\hat{\beta}_{NSRTPE}(k,d)\right)B^{'}\left(\hat{\beta}_{NSRTPE}(k,d)\right)
\end{aligned} \tag{3.8}$$

Now, one can state the following theorem.

Theorem 3.6 The ME is superior to NSRTPE in the mean square error matrix sense if and only if $B^{'}\left(\widehat{\beta}_{NSRTPE}(k,d)\right)D_6^{-1}B\left(\widehat{\beta}_{NSRTPE}(k,d)\right) \le \sigma^2$, where $D_6 = \sigma^2 A - \sigma^2 L_{k,d}AL_{k,d}^{'}$.

Proof: Put $B = P^{'}AP$, with A as defined in Section 1. The matrix is a nonnegative definite matrix and hence so is B.The diagonal elements a_{ij} of B are , therefore , all nonnegative.
Consider

$$tr(A - L_{k,d}AL_{k,d}^{'}) = \sum_{i=1}^{p}\left[a_{ii} - \frac{a_{ii}(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2}\right]$$

Now, in order to prove that $A - L_{k,d}AL_{k,d}^{'} > 0$, we need to show that $a_{ii} - \frac{a_{ii}(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2} > 0$.
Now, we can simplify

$$\begin{aligned}
&a_{ii} - \frac{a_{ii}(\lambda_i+d)^2\lambda_i^2}{(\lambda_i+1)^2(\lambda_i+kd)^2}\\
&= \frac{a_{ij}[kd(\lambda_i+1)+(1-d)\lambda_i][(\lambda_i+1)(\lambda_i+kd)+(\lambda_i+d)\lambda_i]}{(\lambda_i+1)^2(\lambda_i+kd)^2}
\end{aligned}$$

Now, it can be said that the matrix $A - L_{k,d}AL_{k,d}^{'} > 0$. According to Lemma (3), we can say that $MSEM\left(\widehat{\beta}_{ME}\right) - MSEM\left(\widehat{\beta}_{NSRTPE}(k,d)\right) \ge 0$ if and only if $B^{'}\left(\widehat{\beta}_{NSRTPE}(k,d)\right)D_6^{-1}B\left(\widehat{\beta}_{NSRTPE}(k,d)\right) \le \sigma^2$.
This completes the proof.

4. Numerical Illustration

4.1. Real-world example

In order to show the superiority of the proposed estimator, we consider the data set on Total National Research and Development Expenditures as a percent of Gross National product originally due to Gruber (1998), and later considered by Akdeniz and Erol (2003), Li and Yang (2011), Yang and Cui (2011) and Alheety and Kibria (2013). The data set is given as follows:

$$X = \begin{pmatrix} 1.9 & 2.2 & 1.9 & 3.7\\ 1.8 & 2.2 & 2.0 & 3.8\\ 1.8 & 2.4 & 2.1 & 3.6\\ 1.8 & 2.4 & 2.2 & 3.8\\ 2.0 & 2.5 & 2.3 & 3.8\\ 2.1 & 2.6 & 2.4 & 3.7\\ 2.1 & 2.6 & 2.6 & 3.8\\ 2.2 & 2.6 & 2.6 & 4.0\\ 2.3 & 2.8 & 2.8 & 3.7\\ 2.3 & 2.7 & 2.8 & 3.8 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 2.3\\ 2.2\\ 2.2\\ 2.3\\ 2.4\\ 2.5\\ 2.6\\ 2.6\\ 2.7\\ 2.7 \end{pmatrix}$$

Firstly, we obtain the eigenvalue of $X^{'}X$ as $\lambda_1 = 302.963, \lambda_2 = 0.728, \lambda_3 = 0.045, \lambda_4 = 0.035$. And the

Table 1: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.1$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1247	0.1183	0.1112	0.1034	0.0949	0.0858	0.0763	0.0671	0.0601	0.0601	0.0663
LE	0.1881	0.1881	0.1881	0.1881	0.1881	0.1881	0.1881	0.1881	0.1881	0.1881	0.1881
AULE	0.1407	0.1407	0.1407	0.1407	0.1407	0.1407	0.1407	0.1407	0.1407	0.1407	0.1407
MNTPE	0.1986	0.1977	0.1968	0.1959	0.1949	0.1938	0.1928	0.1917	0.1906	0.1894	0.1888
ME	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427
SRLE	0.1874	0.1874	0.1874	0.1874	0.1874	0.1874	0.1874	0.1874	0.1874	0.1874	0.1874
NSRTPE	0.1982	0.1973	0.1963	0.1954	0.1943	0.1933	0.1922	0.1911	0.1899	0.1887	0.1881

Table 2: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.5$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1247	0.1183	0.1112	0.1034	0.0949	0.0858	0.0763	0.0671	0.0601	0.0601	0.0663
LE	0.0797	0.0797	0.0797	0.0797	0.0797	0.0797	0.0797	0.0797	0.0797	0.0797	0.0797
AULE	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611
MNTPE	0.1460	0.1415	0.1366	0.1313	0.1255	0.1192	0.1123	0.1048	0.0968	0.0883	0.0840
ME	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427
SRLE	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
NSRTPE	0.1439	0.1392	0.1340	0.1284	0.1222	0.1153	0.1078	0.0994	0.0902	0.0802	0.0749

Table 3: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.9$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1247	0.1183	0.1112	0.1034	0.0949	0.0858	0.0763	0.0671	0.0601	0.0601	0.0663
LE	0.0684	0.0684	0.0684	0.0684	0.0684	0.0684	0.0684	0.0684	0.0684	0.0684	0.0684
AULE	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793
MNTPE	0.1275	0.1213	0.1144	0.107	0.0988	0.09	0.0807	0.0713	0.0632	0.0594	0.0613
ME	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427
SRLE	0.0373	0.0373	0.0373	0.0373	0.0373	0.0373	0.0373	0.0373	0.0373	0.0373	0.0373
NSRTPE	0.1245	0.1179	0.1105	0.1023	0.0932	0.0832	0.0721	0.0603	0.0485	0.0387	**0.0363**

Table 4: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.99$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1247	0.1183	0.1112	0.1034	0.0949	0.0858	0.0763	0.0671	0.0601	0.0601	0.0663
LE	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793
AULE	0.0808	0.0808	0.0808	0.0808	0.0808	0.0808	0.0808	0.0808	0.0808	0.0808	0.0808
MNTPE	0.1250	0.1185	0.1115	0.1037	0.0952	0.0862	0.0767	0.0675	0.0604	0.0599	0.0656
ME	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427
SRLE	0.0419	0.0419	0.0419	0.0419	0.0419	0.0419	0.0419	0.0419	0.0419	0.0419	0.0419
NSRTPE	0.1219	0.1150	0.1073	0.0987	0.0892	0.0787	0.0673	0.0553	0.0438	**0.0361**	**0.0361**

conditional number is $k = \dfrac{302.963}{0.035} = 8656.086$. This information indicates severe multicollinearity among the explanatory variables.

We consider the following stochastic restriction according to Yildiz (2019):

$$h = H\beta + \upsilon, \quad H = (1, -1, 1, 0), \quad \upsilon \sim N\left(0, \sigma^2\Omega\right)$$

Also, it was obtained that the scalar mean square error (SMSE) of OLSE is 0.0808.

Tables 1-4 were obtained using the estimated SMSE values of RE, LE, AULE, MNTPE, ME, SRLE, and NSRTPE for different k and d values.

From Table 1-4, we can see that the NSRTPE outperforms the other estimators when the parameter k is relatively small and the parameter d is large. However, the NSTRTPE is worse than some existing estimators when d is small.

4.2. Simulation Study

To further illustrate the behavior of the proposed estimator, we perform a Monte Carlo simulation study by considering different levels of multicollinearity. Following McDonald and Galarneau (1975), we generate explanatory variables as follows:

$$x_{ij} = \left(1-\gamma^2\right)^{\frac{1}{2}} z_{ij} + \gamma z_{i,p+1}, \quad \begin{array}{l} i=1,2,\dots,n \\ j=1,2,\dots p \end{array} \tag{4.1}$$

where z_{ij} is an independent normal pseudorandom number and γ is specified so that the theoretical correlation between any two explanatory variables is given by γ^2.

A dependent variable is generated by using the following equation

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \epsilon_i, \quad i=1,2...n \tag{4.2}$$

where ϵ_i is a normal pseudo and variance σ_i^2.

Newhouse and Oman (1971) have noted that if MSEM is a function of σ_i^2 and β, and if the explanatory variables are fixed, the subject to the constraint $\beta\beta^{'} = 1$, the MSEM is minimized when β is the normalized eigenvector corresponding to the largest eigenvalue of the $XX^{'}$ matrix. In this study, we choose the normalized eigenvector corresponding to the largest eigenvalue of the $XX^{'}$ matrix as the coefficient vector, $n = 100, p = 4$ and $\sigma_i^2 = 1$. Three different sets of correlations are considered by selecting the values as $\gamma = 0.7, 0.8$ and 0.9. The estimated SMSE of OLSE at $\gamma = 0.7, 0.8$ and 0.9 are 0.0622, 0.0853 and 0.1572, respectively. In the simulation study, we have used the same stochastic restriction used in Section 4.1.

Tables 5-16 (see Appendix) are obtained using the estimated scalar mean square error (SMSE) values of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE for different d and k values with different sets of correlation, namely, $\gamma = 0.7, 0.8$ and 0.9.According to Table 7, It is observed that the proposed estimator has the smallest scalar mean square error value than other estimators for $k = 0.01$ and 0.005 when $d = 0.9$ and $\gamma = 0.7$. Also, from Table 8, it can be seen that the NSRTPE has the smallest SMSE than other estimators for $k \leq 0.03$. When $d = 0.99$ and $\gamma = 0.7$.From Table 11, one can say that the NSRTPE has the smallest SMSE than other estimators for $k = 0.01$ and 0.005 when $d = 0.9$ and $\gamma = 0.8$. Moreover, Table 12 shows that the NSRTPE showed a better performance for $k \leq 0.04$ when $d = 0.99$ and $\gamma = 0.8$ Based on table 15, one can conclude that the proposed estimator has lower SMSE values compared to those of OLSE, RE, LE, AULE, MNTPE, ME and SRLE for $k = 0.01$ and $k = 0.05$ when $d = 0.9$ and $\gamma = 0.9$. Moreover, Table 16 shows that the proposed estimator has the smallest SMSE than other estimators for $k \leq 0.05$ when $d = 0.99$ and $\gamma = 0.9$. Furthermore, it can be observed in all tables that the NSRTPE is always superior to MNTPE. Nevertheless, the NSTRTPE is worse than some existing estimators when d is small.

5. Conclusion

A new biased estimator has been proposed for estimating the parameter of the multiple linear regression with multicollinearity when the stochastic restrictions are available. Moreover, necessary and sufficient conditions for the superiority of the proposed estimator over the OLSE, RE, LE, AULE, MNTPE, ME, SRLE in the MSEM sense have been discussed. Finally, we illustrated our findings with a real-world example and a Monte Carlo simulation. From the results of the real-world example and simulation study, it could be concluded that the proposed estimator performs well compared to others when the shrinkage parameters k and d are relatively small and large, respectively.

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Appendix

Lemma 1: Let $n \times n$ matrices $M > 0, N > 0$ $(or N \geq 0$), then $M > N$ if and only if $\lambda_1 \left(NM^{-1}\right) < 1$, where $\lambda_1 \left(NM^{-1}\right)$ is the largest eigenvalue of the matrix NM^{-1} (Wang, 2006).

Lemma2: Let $\hat{\beta_1}$ and $\hat{\beta_2}$ be two linear estimators of β. Suppose that $D = D\left(\hat{\beta_1}\right) - D\left(\hat{\beta_2}\right)$ is positive definite then $\Delta = MSEM\left(\hat{\beta_1}\right) - MSEM\left(\hat{\beta_2}\right)$ is nonnegative definite if and only if $b_2^{'}\left(D + b_1 b_1^{'}\right)^{-1} b_2 \leq 1$, where b_j denotes the bias vector of $\hat{\beta_j},\ j = 1, 2$ (Trenkler and Toutenburg, 1990).

Lemma 3 : Let M be a positive definite matrix, namely $M > 0$, α be some vector, then $M - \alpha\alpha^{'} \geq 0$ if and only if $\alpha^{'} M^{-1}\alpha \leq 1$ (Farebrother, 1976).

Table 5: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.1$ and $\gamma = 0.7$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1447	0.1287	0.1129	0.0976	0.0829	0.0695	0.0578	0.0488	0.0441	0.0465	0.0521
LE	0.6372	0.6372	0.6372	0.6372	0.6372	0.6372	0.6372	0.6372	0.6372	0.6372	0.6372
AULE	0.3657	0.3657	0.3657	0.3657	0.3657	0.3657	0.3657	0.3657	0.3657	0.3657	0.3657
MNTPE	0.6579	0.6559	0.6539	0.6519	0.6499	0.6478	0.6457	0.6436	0.6415	0.6394	0.6383
ME	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451
SRLE	0.6369	0.6369	0.6369	0.6369	0.6369	0.6369	0.6369	0.6369	0.6369	0.6369	0.6369
NSRTPE	0.6577	0.6557	0.6537	0.6516	0.6496	0.6475	0.6454	0.6433	0.6412	0.6390	0.6380

Table 6: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.5$ and $\gamma = 0.7$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1447	0.1287	0.1129	0.0976	0.0829	0.0695	0.0578	0.0488	0.0441	0.0465	0.0521
LE	0.2135	0.2135	0.2135	0.2135	0.2135	0.2135	0.2135	0.2135	0.2135	0.2135	0.2135
AULE	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718
MNTPE	0.3327	0.3216	0.3102	0.2985	0.2867	0.2746	0.2624	0.2500	0.2377	0.2254	0.2194
ME	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451
SRLE	0.2088	0.2088	0.2088	0.2088	0.2088	0.2088	0.2088	0.2088	0.2088	0.2088	0.2088
NSRTPE	0.3315	0.3202	0.3087	0.2969	0.2848	0.2725	0.2600	0.2472	0.2344	0.2215	0.2151

Table 7: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.9$ and $\gamma = 0.7$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1447	0.1287	0.1129	0.0976	0.0829	0.0695	0.0578	0.0488	0.0441	0.0465	0.0521
LE	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588
AULE	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611
MNTPE	0.1708	0.155	0.1393	0.1237	0.1085	0.0939	0.0804	0.0685	0.0594	0.0549	0.0554
ME	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451
SRLE	0.0448	0.0448	0.0448	0.0448	0.0448	0.0448	0.0448	0.0448	0.0448	0.0448	0.0448
NSRTPE	0.1689	0.1529	0.1369	0.1209	0.1052	0.0899	0.0755	0.0625	0.0518	**0.0448**	**0.0436**

Table 8: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.99$ and $\gamma = 0.7$

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1447	0.1287	0.1129	0.0976	0.0829	0.0695	0.0578	0.0488	0.0441	0.0465	0.0521
LE	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611	0.0611
AULE	0.0621	0.0621	0.0621	0.0621	0.0621	0.0621	0.0621	0.0621	0.0621	0.0621	0.0621
MNTPE	0.1471	0.1311	0.1153	0.0999	0.0852	0.0716	0.0597	0.0503	0.0451	0.0468	0.0518
ME	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451	0.0451
SRLE	0.0443	0.0443	0.0443	0.0443	0.0443	0.0443	0.0443	0.0443	0.0443	0.0443	0.0443
NSRTPE	0.1451	0.1288	0.1127	0.0969	0.0816	0.0672	0.0543	**0.0436**	**0.0364**	**0.0351**	**0.0379**

Table 9: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.1$ and γ=0.8

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1377	0.1234	0.1094	0.0958	0.0828	0.0708	0.0605	0.0528	0.0498	0.0561	0.0662
LE	0.6264	0.6264	0.6264	0.6264	0.6264	0.6264	0.6264	0.6264	0.6264	0.6264	0.6264
AULE	0.3460	0.3460	0.3460	0.3460	0.3460	0.3460	0.3460	0.3460	0.3460	0.3460	0.3460
MNTPE	0.6468	0.6448	0.6429	0.6409	0.6389	0.6369	0.6349	0.6328	0.6307	0.6286	0.6275
ME	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610
SRLE	0.6260	0.6260	0.6260	0.6260	0.6260	0.6260	0.6260	0.6260	0.6260	0.6260	0.6260
NSRTPE	0.6465	0.6446	0.6426	0.6406	0.6386	0.6366	0.6345	0.6324	0.6303	0.6282	0.6271

Table 10: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.5$ and γ=0.8

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1377	0.1234	0.1094	0.0958	0.0828	0.0708	0.0605	0.0528	0.0498	0.0561	0.0662
LE	0.2159	0.2159	0.2159	0.2159	0.2159	0.2159	0.2159	0.2159	0.2159	0.2159	0.2159
AULE	0.0829	0.0829	0.0829	0.0829	0.0829	0.0829	0.0829	0.0829	0.0829	0.0829	0.0829
MNTPE	0.3249	0.3145	0.3039	0.2931	0.2821	0.2708	0.2595	0.2480	0.2367	0.2257	0.2206
ME	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610
SRLE	0.2094	0.2094	0.2094	0.2094	0.2094	0.2094	0.2094	0.2094	0.2094	0.2094	0.2094
NSRTPE	0.3237	0.3132	0.3024	0.2914	0.2801	0.2686	0.2568	0.2448	0.2327	0.2207	0.2149

Table 11: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.9$ and γ=0.8

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1377	0.1234	0.1094	0.0958	0.0828	0.0708	0.0605	0.0528	0.0498	0.0561	0.0662
LE	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774
AULE	0.0838	0.0838	0.0838	0.0838	0.0838	0.0838	0.0838	0.0838	0.0838	0.0838	0.0838
MNTPE	0.1633	0.1491	0.1349	0.1209	0.1072	0.0940	0.0819	0.0714	0.0640	0.0631	0.0673
ME	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610
SRLE	0.0576	0.0576	0.0576	0.0576	0.0576	0.0576	0.0576	0.0576	0.0576	0.0576	0.0576
NSRTPE	0.1616	0.1472	0.1327	0.1182	0.104	0.0901	0.0769	0.0649	0.0553	**0.0505**	**0.0517**

Table 12: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.99$ and γ=0.8

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1377	0.1234	0.1094	0.0958	0.0828	0.0708	0.0605	0.0528	0.0498	0.0561	0.0662
LE	0.0837	0.0837	0.0837	0.0837	0.0837	0.0837	0.0837	0.0837	0.0837	0.0837	0.0837
AULE	0.0853	0.0853	0.0853	0.0853	0.0853	0.0853	0.0853	0.0853	0.0853	0.0853	0.0853
MNTPE	0.1401	0.1258	0.1117	0.0980	0.0850	0.0729	0.0623	0.0543	0.0507	0.0562	0.0657
ME	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610	0.0610
SRLE	0.0599	0.0599	0.0599	0.0599	0.0599	0.0599	0.0599	0.0599	0.0599	0.0599	0.0599
NSRTPE	0.1383	0.1238	0.1093	0.0952	0.0815	0.0686	**0.0568**	**0.0471**	**0.0408**	**0.0417**	**0.0474**

Table 13: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.1$ and γ=0.9

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1721	0.1589	0.1457	0.1324	0.1193	0.1064	0.0941	0.0830	0.0757	0.0818	0.1015
LE	0.6435	0.6435	0.6435	0.6435	0.6435	0.6435	0.6435	0.6435	0.6435	0.6435	0.6435
AULE	0.3518	0.3518	0.3518	0.3518	0.3518	0.3518	0.3518	0.3518	0.3518	0.3518	0.3518
MNTPE	0.6676	0.6655	0.6633	0.6611	0.6588	0.6564	0.654	0.6515	0.6489	0.6463	0.6449
ME	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
SRLE	0.6428	0.6428	0.6428	0.6428	0.6428	0.6428	0.6428	0.6428	0.6428	0.6428	0.6428
NSRTPE	0.6673	0.6652	0.6630	0.6607	0.6584	0.6560	0.6536	0.6510	0.6484	0.6457	0.6443

Table 14: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.5$ and γ=0.9

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1721	0.1589	0.1457	0.1324	0.1193	0.1064	0.0941	0.0830	0.0757	0.0818	0.1015
LE	0.2389	0.2389	0.2389	0.2389	0.2389	0.2389	0.2389	0.2389	0.2389	0.2389	0.2389
AULE	0.1236	0.1236	0.1236	0.1236	0.1236	0.1236	0.1236	0.1236	0.1236	0.1236	0.1236
MNTPE	0.3541	0.3439	0.3333	0.3223	0.3109	0.2989	0.2865	0.2735	0.2601	0.2474	0.2422
ME	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
SRLE	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269
NSRTPE	0.3531	0.3428	0.3320	0.3208	0.3091	0.2967	0.2836	0.2697	0.2550	0.2400	0.2329

Table 15: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.9$ and γ=0.9

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1721	0.1589	0.1457	0.1324	0.1193	0.1064	0.0941	0.0830	0.0757	0.0818	0.1015
LE	0.1359	0.1359	0.1359	0.1359	0.1359	0.1359	0.1359	0.1359	0.1359	0.1359	0.1359
AULE	0.1543	0.1543	0.1543	0.1543	0.1543	0.1543	0.1543	0.1543	0.1543	0.1543	0.1543
MNTPE	0.1966	0.1832	0.1697	0.1560	0.1421	0.1281	0.1142	0.1009	0.0901	0.0890	0.1003
ME	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
SRLE	0.0983	0.0983	0.0983	0.0983	0.0983	0.0983	0.0983	0.0983	0.0983	0.0983	0.0983
NSRTPE	0.1954	0.1818	0.1680	0.1539	0.1395	0.1247	0.1096	0.0945	0.0802	**0.0718**	**0.0758**

Table 16: The estimated SMSE of RE, LE, AULE, MNTPE, ME, SRLE and NSRTPE when $d = 0.99$ and γ =0.9

k	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
RE	0.1721	0.1589	0.1457	0.1324	0.1193	0.1064	0.0941	0.0830	0.0757	0.0818	0.1015
LE	0.1542	0.1542	0.1542	0.1542	0.1359	0.1542	0.1542	0.1542	0.1542	0.1542	0.1542
AULE	0.1572	0.1572	0.1572	0.1572	0.1543	0.1572	0.1572	0.1572	0.1572	0.1572	0.1572
MNTPE	0.1744	0.1611	0.1478	0.1345	0.1421	0.1083	0.0957	0.0844	0.0767	0.0820	0.1007
ME	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
SRLE	0.1090	0.1090	0.1090	0.1090	0.1090	0.1090	0.1090	0.1090	0.1090	0.1090	0.1090
NSRTPE	0.1731	0.1597	0.1461	0.1324	0.1395	**0.1047**	**0.0909**	**0.0774**	**0.0658**	**0.0625**	**0.0724**