

SOME BEHAVIOR OF COARSE STRUCTURE AND COARSE EQUIVALENCE

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ABSTRACT. I am going to introduce some properties of coarse structure. Coarse space is defined for large scale in metric space similar to the tools provided by topology for analyzing behavior at small distance, as topological property can be defined entirely in terms of open sets. Analogously a large scale property can be defined entirely in terms of controlled sets. The properties we required were that the maps were coarse (proper and bornologous), but why do these maps imply that the spaces have the same large structure? Essentially this has to do with contractibility. Spaces which are the same on a large scale can be scaled so that the points are not too far away from each other, but we are not concerned with any differences on small scale that may arise. In addition I am going to explain some basic definition related with the title of my research work besides ,I want to investigate several results in coarse map, coarse equivalent and coarse embedding. Further I have to proof some results of product of coarse structure.

Coarse map need not be a continuous map. Coarse space has some application in various parts in mathematics. More over coarse structure is a large scale property so we can invest some results related with coarse space and topology. Topology is the small scale structure, but topological coarse structure is the large scale structure. We investigated some results about coarse maps, coarse equivalent and coarse embedding.

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1. INTRODUCTION

Imagine a new student of analysis. In calculus he hears about limit and continuity, probably at first in a quite informal way: The limit is what happens on the small scale, later this idea formalized in terms of the classical definition and soon it becomes apparent that the no tune domain of this definition is the world of metric spaces. Then perhaps in the first graduate course, the student takes the final step in this journey of abstraction. He learns that what really matters in understanding limits and continuity is not numerical value of the metric, just the open sets that it defined. This realization leads naturally to the abstract notation at topological space but it also enhances understanding even in the metrizable world for in stone, there is only one natural topology on a finite- dimensional (real) vector space, though there are many matrices that give rise to it.

The notation of coarse space arises through a similar process of abstraction starting with the informal idea of studying what happens on the large scale to understand this idea, consider the metric spaces \mathbb{Z}^n and \mathbb{R}^n . Their small scale structure their topology is entirely different, but the large scale they resemble each other closely. Any geometric configuration in \mathbb{R}^n can be approximated by one in \mathbb{Z}^n to within in a uniformly bounded error.

We think such spaces as coarse equivalent. Formally speaking, a coarse structure on a set X is defined to be a collection of subsets of $X \times X$ called controlled set entourages for the coarse structure which satisfy some simple axioms. It is more accurate to say that a coarse structure is the large scale counter part of a uniformly then of a topology.

Coarse structures are on abstract construction describing of a space at a large distance. In journal variety of existing results on coarse structure are presented, with the main focus being coarse embedding into coarse equivalent. Coarse structure defined, and it is shown how a metric induces a coarse structure, coarse maps, coarse equivalent and embedding are defined, and some of their basis properties are presented.

Coarse spaces are sets equipped with a coarse structure, which describes the behaviour of the space at the distance. A coarse space has a well define notion of boundedness and bounded subsets. One can obtain some intuition on the concept by considering a extremely zoomed-out view of a space, under which for example the spaces and \mathbb{R} look similar.

To study the large scale structure of the metric space or topological space ,one naturally becomes concerned with properties which hold at large scales, such as boundedness, degrees of freedom and restriction of movement :continuity becomes correspondingly less important ,since it has little impact on these qualities of a space. Coarse geometry provides a set of tools for discussion of large scale structure by consideration of maps which preserve these properties.

2. PRELIMINARIES

Definition 2.1. Let X be a set, $\epsilon \subset \mathbb{P}(X \times X)$, the collection ϵ is a coarse structure on X . Then the following conditions apply

- (1) *Diagonal property:* Δ_X is an element of ϵ , i.e $\Delta_X = \{(x, x) : (x, x) \in X\}$
- (2) *Subset property:* $A \in \epsilon, B \subseteq A$ then $B \in \epsilon$;
- (3) *Finite union property:* $A \in \epsilon, B \in \epsilon$ then $A \cup B \in \epsilon$;
- (4) *Inverse property:* $A^{-1} = \{(y, x) \in X^2 : (x, y) \in A\}$;
- (5) *Composition property:* $A, B \subseteq X^2$,

$$A \circ B = \{(x, z) \in X^2, \exists y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\} .$$

The elements of the coarse space commonly referred to as controlled sets or entourages. Then the pair (X, ϵ) is called coarse space.

Definition 2.2. [4] Let X be a set and ϵ be a coarse structure on X . let B be a non empty subset of X . we say that X is bounded with respect to ϵ , if there is an $x \in X$ such that $B \times \{x\}$ is controlled.

Definition 2.3. [1] (Coarse Map) Let X, Y be a metric spaces and $f : X \rightarrow Y$ be a map. The map f is called coarsely proper (metrically proper), if for every bounded set B in Y then $f^{-1}(B)$ is bounded in X . If f is called bornologous (uniformly) if $\forall R > 0, \exists S > 0$ Such that $d_X(x, y) < R$ then $d_Y(f(x), f(y)) < S$. Then f is called coarse map if it is both coarsely proper and bornologous.

Definition 2.4. [1] (Closeness) Let S be a set and X be a coarse space then two maps $f, g : S \rightarrow X$ is called close if $f(s), g(s) \subseteq X^2$ is controlled set. In particular metric space $\{d(f(s), g(s)); s \in X\}$ is bounded set.that is

$$\sup \{d(f(s), g(s)) \mid s \in X\} < \infty.$$

Definition 2.5. [1] (Coarse Embedding) Let X, Y be a metric spaces . A (not necessarily continuous) function $f : X \rightarrow Y$ is a coarse embedding if there exists non decreasing functions $\rho_1, \rho_2 : [0, \infty) \rightarrow [0, \infty)$ satisfying

- (1) $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$ for all $x, y \in X$;
- (2) $\lim_{t \rightarrow \infty} \rho_1(t) = +\infty$.

Definition 2.6. [1] (Coarse Equivalent) We say that metric space X and Y are coarse equivalent if there exists coarse maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g$ and $g \circ f$ are closed to the identity maps on Y and X . Either of the maps f, g is then called a coarse equivalent.

3. EXAMPLES

Example 1. Let (X, d) be a metric space. Define ϵ_d , as follows $\epsilon_d = \{E \subseteq X^2 : d[E] < \infty\}$. Then ϵ_d is a coarse structure, where $d[E] = \sup_{x, y \in X} d(x, y)$. The coarse structure ϵ_d is called bounded coarse structure of (X, d) . Since $d(x, x) = 0$, $(x, x) \in \Delta_X$. Consider $d[\Delta_X] = \sup_{x, x \in X} d(x, x) = 0 < \infty$. Therefore $\Delta_X \in \epsilon_d$. Let $A \in \epsilon_d, d(x, y) = d(y, x) : (x, y) \in A$ implies that $d[A^{-1}] = \sup_{x, y \in X} d(y, x) d[A^{-1}] = \sup_{x, y \in A} d(x, y); \forall (x, y) \in A$ therefore $d[A^{-1}] < \infty$. Therefore $A^{-1} \in \epsilon_d$. Let B be subset of A , Let $(x, y) \in B$ and $d[B] = \sup_{x, y \in B} d(x, y) = \sup_{x, y \in A} d(x, y) = d[A] < \infty$, hence $d[B] < \infty$. Therefore $B \in \epsilon_d$. Let $A, B \in \epsilon_d$. So $d[A], d[B] < \infty$. Let $(a, b) \in A \cup B$. Then $(a, b) \in A \vee (a, b) \in B$. Therefore $d(a, b) \leq d[A] \vee d[B] d(a, b) \leq \max\{d[A], d[B]\}$ implies that $d(a, b) \leq d[A \cup B] \leq \max\{d[A], d[B]\} < \infty$, therefore $d[A \cup B] < \infty$. Therefore $A \cup B \in \epsilon_d$. Let $(a, c) \in A \circ B$. There exist $b \in X$ Such that $(a, b) \in A$ and $(b, c) \in B$. Then $d(a, c) \leq d(a, b) + d(b, c). d[A] + d[B] < \infty$. Then $d[A \circ B] \leq d[A] + d[B] < \infty$ so $d[A \circ B] < \infty$. Hence $A \circ B \in \epsilon_d$. Therefore ϵ_d is a bounded coarse structure.

Example 2. Let $f : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ be defined as $f(x) = \sqrt{x}$. Show that f is coarse map. First we shall show that f is coarse map. For bornologous, assume that $|x - y| < R$. For a Case I: $x, y \in [0, 1]$. Then $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left| \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} \right| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq 1$. For a Case II: If $x \geq 1, y \geq 1$, $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left| \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} \right| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq R$. For Case III: If $x \geq 1, y < 1$ $\frac{1}{\sqrt{x} + \sqrt{y}} \leq 1$ implies that $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left| \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} \right| =$

$\left| \frac{x-y}{\sqrt{x}+\sqrt{y}} \right| \leq R$. Therefore f is uniformly (bornologous). For metrically (proper) Let S be a bounded subset of $\mathbb{R}^+ \cup \{0\}$. There exist an interval (a, b) such that $S \subseteq (a, b)$, since f is increasing function, $f^{-1}(S) \subseteq (a^2, b^2)$. Which is bounded so f is metrically (proper). Therefore f is coarse map.

Example 3. Let (X, d_X) be a bounded metric space. Let $Y = ([0, M], |\cdot|)$, where $M = \sup_{x,y \in X} d(x, y)$. Then X is coarse equivalent to Y . Let $x, y \in X$. Define $F : X \rightarrow Y$ by $F(x) = d_X(x, x_0)$ and $G : Y \rightarrow X$ by $G(x) = y_0$, we have to prove that F is coarse map, G is coarse map, $F \circ G$ is closed to identity of X and $G \circ F$ is closed to identity of Y . For let S be a bounded subset of Y . Consider $F^{-1}(S)$, since (X, d) be a bounded metric space. $F^{-1}(S) \subseteq X$ so $F^{-1}(S)$ is also bounded. So f is proper. Now let $x, y \in X$ satisfy $d_X(x, y) < R$. Consider $|F(x) - F(y)| = |d_X(x, y_0) - d_X(y, y_0)| \leq M$. Since F is bornologous. Therefore F is coarse map. Now consider $G : Y \rightarrow X$, let S be a bounded subset of X . Consider $G^{-1}(S)$, $|G^{-1}(S)| \leq M$ so $G^{-1}(S)$ is bounded. Therefore G is proper. Let $y_1, y_2 \in Y$, $|G(y_1) - G(y_2)| = |y_0 - y_0| = 0 \leq K, \forall K > 0$. Therefore G is bornologous. Hence G is coarse map. Consider $d_Y(F \circ G, x) = |d_X(y_0, y_0) - x| = x < M$ and also $d_X(G \circ F, x) = d_X(y_0, x) \leq \sup_{x,y \in X} d_X(x, y) = M$.

Thus X and Y are coarse equivalent.

Example 4. Let (X, d_X) be a bounded metric space. Let $Y = ([0, M], |\cdot|)$, where $M = \sup_{x,y \in X} d(x, y)$. Then X is coarse equivalent to Y . Let $x, y \in X$. Define $F : X \rightarrow Y$ by $F(x) = d_X(x, x_0)$ and $G : Y \rightarrow X$ by $G(x) = y_0$. We shall show that F is coarse map, G is coarse map, $F \circ G$ is closed to identity of X , and $G \circ F$ is closed to identity of Y . Let S be a bounded subset of Y . Consider $F^{-1}(S)$, since (X, d) be a bounded metric space. $F^{-1}(S) \subseteq X$ so $F^{-1}(S)$ is also bounded. So f is proper. Now let $x, y \in X$ satisfy $d_X(x, y) < R$. Consider $|F(x) - F(y)| = |d_X(x, y_0) - d_X(y, y_0)| \leq M$. Since F is bornologous. Therefore F is coarse map. Now consider $G : Y \rightarrow X$, let S be a bounded subset of X . Consider $G^{-1}(S)$, $|G^{-1}(S)| \leq M$ so $G^{-1}(S)$ is bounded. Therefore G is proper. Let $y_1, y_2 \in Y$, $|G(y_1) - G(y_2)| = |y_0 - y_0| = 0 \leq K, \forall K > 0$. Therefore G is bornologous. Therefore G is coarse map. Consider $d_Y(F \circ G, x) = |d_X(y_0, y_0) - x| = x \leq M$, and also $d_X(G \circ F, x) = d_X(y_0, x) \leq \sup_{x,y \in X} d_X(x, y) = M$. Thus X and Y are coarse equivalent.

Proposition 3.1. *Let $h : X \rightarrow Y$ be a coarse embedding, then X and $h(X)$ are coarse equivalent.*

Proof. Let $f : X \rightarrow h(X)$ defined by $f(x) = h(x)$, $\forall x \in X$. Let take ρ_1, ρ_2 be non-decreasing functions. Clearly f is onto (because h is onto). First we shall show that f is coarse. For bornologous (uniformly), let $R > 0$, suppose that $d_X(x, y) < R$. Since ρ_2 is non-decreasing $\rho_2(d_X(x, y)) < \rho_2(R)$. Since f is coarse embedding we have $d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y)) < \rho_2(R)$. Hence f is bornologous. For f is proper, let $S \subset h(X)$ be bounded $\forall y, y' \in S$ there exists $R \geq 0$ such that $d_Y(y, y') \leq R$. Consider $f^{-1}(S)$, let $x, x' \in f^{-1}(S)$, there exists $y, y' \in S$ such that $y = f(x), y' = f(x')$. Since f is coarse embedding we have $\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) = d_Y(y, y') \leq R$. Let $R' = \sup \{\rho_1^{-1}(\{R\})\}$, Where $\rho_1^{-1}(\{R\})$ is the pre-image of the singleton set $\{R\}$ under ρ_1 . Since $\lim_{t \rightarrow \infty} (\rho_1)(t) = +\infty$ and ρ_1 is non-decreasing then R' is finite. Thus $d_X(x, x') \leq R'$. Therefore f is proper this shows that f is coarse map. Now we define the function $g : f(X) \rightarrow X$, for every $y \in f(X)$. We choose $g(y)$ to be an element $x \in X$ such that $f(x) = y$. We can now show that g is coarse. We begin by showing g is bornologous. Let $R > 0$, assume that $d_Y(y, y') < R$, there exists $x, x' \in X$ such that $f(x) = y, f(x') = y'$, where $g(y) = x, g(y') = x'$. Since $y, y' \in f(X)$, since f is coarse embedding $\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) = d_Y(y, y') < R$. By a similar argument as before let $R' = \sup \{\rho_1^{-1}(\{R\})\}$. We see that $R' = \sup \{\rho_1^{-1}(\{R\})\}$. Proving that g is bornologous. To show g is proper. let $S \subset X$ be bounded $\forall x, x' \in S$ there exists $R > 0$. Such that $d_Y(y, y') \leq R$. Consider the set $f^{-1}(S)$. Let $y, y' \in f^{-1}(S)$, there exists $x, x' \in S$ such that $x = g(y), x' = g(y')$ by definition of g we have $f(x) = y, f(x') = y'$. Since f is coarse embedding $d_Y(y, y') = d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x')) \leq \rho_2(R)$. Which is shows that $f^{-1}(S)$ is bounded. Thus g is proper. Therefore g is coarse. To show $f \circ g$ is closed to the identity in X . $d_Y(f \circ g(y), y) = d_Y(f(g(y)), y) = d_Y(f(x), y) = d_Y(y, y) = 0$. Now $g \circ f$ is closed to identity in X . By the definition of $g, \exists x' \in X$ such that $g(f(x)) = x'$, where $f(x) = f(x')$, we can rewrite $d_X(g(f(x)), x) = d_X(x', x)$, and f is coarse embedding. We have $\rho_1(d_X(x', x)) \leq d_Y(f(x), f(x')) = 0$. Since $f(x) = f(x')$. Letting $C' = \sup \{\rho_1^{-1}(\{0\})\}$ we see that $d_X(x', x) \leq C'$, we thus have shown that for all $x \in X$. $d_X((g \circ f)(x), x) = d_X(g(f(x)), x) = d_X(x', x) \leq C'$. \square

Proposition 3.2. *Let X and Y be metric spaces and say that X is coarsely equivalent to Y . Then coarse equivalent is an equivalence relation.*

Proof. Clearly X coarse equivalent to X , letting f and g each be the identity maps. id_X on X , so since identity map is coarse map means that $f \circ g$ and $g \circ f$ are actually identity maps from X to X . So X coarse equivalent to X is reflexive. If X is closed to Y then by the symmetric of the definition of coarse equivalence, it follows that Y is closed to X . Thus symmetric of coarse equivalent is satisfied. Suppose that X coarse equivalent to Y and Y coarse equivalent to Z . Then there are maps $f_X : X \rightarrow Y$, $g_Y : Y \rightarrow X$, $f_Y : Y \rightarrow Z$ and $g_Z : Z \rightarrow Y$ such that $f_X \circ g_Y$ and $g_Y \circ f_X$ are close to the identity maps id_Y and id_X on Y and X respectively, and $f_Y \circ g_Z$ and $g_Z \circ f_Y$ are close to the identity maps id_Z and id_Y on Z and Y respectively. Since composition of coarse map is coarse map $f_Y \circ f_X : X \rightarrow Z$ and $g_Y \circ g_Z : Z \rightarrow X$ are coarse maps, $(f_Y \circ f_X) \circ (g_Y \circ g_Z) = f_Y \circ ((f_X \circ g_Y) \circ g_Z) = f_Y \circ (id_Y \circ g_Z) = f_Y \circ g_Z = id_Z$, and also $(g_Y \circ g_Z) \circ (f_Y \circ f_X) = g_Y \circ ((g_Z \circ f_Y) \circ f_X) = g_Y \circ (id_Y \circ f_X) = g_Y \circ f_X = id_X$. This shows that there exists coarse maps $f = f_Y \circ f_X$ from X to Z and $g = g_Y \circ g_Z$ from Z to X such that $f \circ g$ coarse equivalent to id_Z and $g \circ f$ coarse equivalent to id_X . Thus X coarse equivalent to Y and Y coarse equivalent to Z , so X coarse equivalent to Z so coarse equivalent is transitive. Therefore coarse equivalent is an equivalence relation. \square

Example 5. \mathbb{Z} is coarse equivalent to \mathbb{R} . Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be the inclusion map $x \mapsto x$ and letting $g : \mathbb{R} \rightarrow \mathbb{R}$ be the map $x \mapsto \lfloor x \rfloor$. f is coarse map since it is inclusion map and g is greatest integer function is also coarse map. $(f \circ g)(x) = f(g(x)) = f(\lfloor x \rfloor) = \lfloor x \rfloor$ for each $x \in \mathbb{R}$, $(g \circ f)(x) = g(f(x)) = g(x) = \lfloor x \rfloor = x$ for each $x \in \mathbb{Z}$. $g \circ f$ closed to the identity map on \mathbb{Z} is obvious. (Since it is the identity map) That $f \circ g$ is close to the identity map on \mathbb{R} is also clear, since $d(\lfloor x \rfloor, x) < 1$ for all $x \in \mathbb{R}$. It follows that \mathbb{Z} is coarse equivalent to \mathbb{R} .

Example 6. \mathbb{Q} is coarse equivalent to \mathbb{Z} and hence \mathbb{R} . Let $f : \mathbb{Q} \rightarrow \mathbb{Z}$ be the inclusion map $x \mapsto x$ and letting $g : \mathbb{Z} \rightarrow \mathbb{Q}$ be the map $x \mapsto \lfloor x \rfloor$. f is coarse map since it is inclusion map and a greatest integer function g is also coarse map. $(f \circ g)(x) = f(g(x)) = f(\lfloor x \rfloor) = \lfloor x \rfloor$ for each $x \in \mathbb{Z}$, $(g \circ f)(x) = g(f(x)) = g(x) = \lfloor x \rfloor = x$ for each $x \in \mathbb{Q}$. $g \circ f$ closed to the identity map on \mathbb{Q} is obvious. (Since it is the identity map) That $f \circ g$ is close to the identity map on \mathbb{Z} is clear also, since $d(\lfloor x \rfloor, x) < 1$ for all $x \in \mathbb{Z}$. It follows that \mathbb{Q} is coarse equivalent to \mathbb{Z} . Now \mathbb{Z} is

coarse equivalent to \mathbb{R} and \mathbb{Q} is coarse equivalent to \mathbb{Z} , then \mathbb{Q} is coarse equivalent to \mathbb{R} . Since coarse equivalence is an equivalent relation.

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REFERENCES

- [1] J. ROE: *What is a coarse space?*, Notices of the American Mathematical society, (2006), 668-669.
- [2] E. ELFVING: *Master Thesis*, University of Helsinki, Faculty of science, Department of Mathematics and Statistics.
- [3] A.N. DRANISHNIKOV, J. KEESLING, V.V. USPENSKIJ: *On the Higson Corona of Uniformly Contractible Spaces*, Topology, **37**(4) (2000), 791-803.
- [4] P. JOLISSAINT: *Rapidly decreasing Functions in reduced C^* -algebras of groups*, Transactions of the American Mathematical Society, **317**(1) (1990), 167-196.
- [5] J. DYDAK, C.S. HOFFLAND: *Alternative Definition of Coarse Structures*, arXiv:math/0605562.
- [6] K. KANNAN: *Strong invariant approximation property for Discrete groups*, Int. J. Pure Appl. Math., **85**(6) (2013), 1075-1086.

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