ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.10, 7725–7732
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.3 Spec. Issue on ACMAMP-2020

SOME RESULTS ON FUZZY METRIC SPACE AND SOME EXAMPLES OF FUZZY *b*-METRIC SPACE

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ABSTRACT. The problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several researchers from different point of view. The concept of fuzzy sets was introduced by Zadeh. Following fuzzy metric space and fuzzy $b-$metric space modified by Kramosil, Mickalek-George and Veeramani using continuous triangular norm. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm t-norm, if $*$ is associative, commutative, continuity, monotonicity and 1 acts as identity element. Some typical examples of $t-$norm are product $t-$norm, minimum $t-$norm, lukasiewitz $t-$ norm and hamacher $t-$norm. In our work we used minimum triangular $t-$norm and Banach fixed point theorem to prove fixed point theorem in Fuzzy metric space and discuss some examples of Fuzzy $b-$metric space. Letting $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ is a continuous function satisfying the condition

$$M(Tx, Ty, t) \geq \min\{M(x, Tx, t), M(y, Ty, t), M(x, y, t)\}$$

and $\lim_{t\to\infty} M(x, y, t) = 1$, where $x, y \in X, x \neq y$ and M is a Fuzzy set. We proved T has a fixed point in X. Moreover we proved some examples of fuzzy $b-$ metric space under minimum $t-$norm condition.

1. INTRODUCTION

Metric spaces and their various generalizations occur frequently in computer science applications. The problem of constructing a satisfactory theory of fuzzy

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2020 *Mathematics Subject Classification.* 94D05.

Key words and phrases. $b-$metric space, fuzzy metric space, continuous triangular norm, fuzzy $b-$metric space.

metric spaces has been investigated by several researchers from different point of view. Zadeh invented the theory of fuzzy set in 1965. The concept of fuzzy metric space was introduced initially by Kramosil and Michalek.Later on, George and veeramani gives the modified notation of fuzzy metric space. A useful theory of fixed points in fuzzy metric spaces is established by Grabiec, Dey and Saha recently, the notion of fuzzy b-metric spaces is investigated. Grabiec defined fuzzy Cauchy sequence, fuzzy convergence sequence, fuzzy complete metric space using these Grabiec extended the well-known Banach and Edelsteiný̋ fixed point theorems to fuzzy metric spaces in the senses of Kramosil and Michalek Frang (1992) established some fixed point theorem for contraction type mappings and Mishra et.al (1994) obtained common fixed point theorems for contraction mapping and asymptotically commuting mapping. Later in 1994, A. George and P. Veeramani modified the notion of fuzzy metric space with the help of triangular norm. A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous triangular norm (t-norm), if $*$ is associative, commutative, continuity, monotone and 1 acts as identity element. Some typical examples of t-norm are product t-norm, minimum t-norm, lukasiewitz t- norm and hamacher t-norm.

2. Preliminaries

Definition 2.1. *($b-$ metric space) Let X be a non empty set and $k\geq 1$ be a given real number. A function $d:X\times X\rightarrow[0,\infty)$ is called a $b-$ metric space provided that for all $x,y,z\in X$,*

(1) *$d(x,y)=0$ if and only if $x=y$;*
(2) *$d(x,y)=d(y,x)$;*
(3) *$d(x,z)\leq k\,[d(x,y)+d(y,z)]$.*

A Pair (X,d,k) is called a $b-$ metric space. It is clear that definition of $b-$ metric space is an extension of usual metric space.

Definition 2.2. *((Continuous triangular norm (t-norm)) A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous triangular norm (t- norm) if it is satisfying following condition:*

(1) *$*$ is associative and commutative;*
(2) *$*$ is continuous;*

(3) $a * 1 = a$, $\forall\, a \in [0, 1]$;
(4) $a * b \leq c * d$ *where* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0, 1]$.

Some typical examples of t-norm are the following:

(1) $a * b = ab$ *(Product);*
(2) $a * b = \min\{a, b\}$ *(minimum);*
(3) $a * b = max\{a + b - 1, 0\}$ *(Lukasiewicz);*
(4) $a * b = \frac{ab}{a+b-1}$ *(Hamacher).*

Definition 2.3. *(Fuzzy Metric Space) A 3-tuple $(X, M, *)$ is called fuzzy metric space if X is an arbitrary set $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$,*

(1) $M(x, y, 0) = 0$;
(2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous and* $\lim_{t\to\infty} M(x, y, t) = 1$.

Definition 2.4. *(Fuzzy $b-$ Metric space) Let X be a non empty set, let $k \geq 1$ be a given real number and $*$ be a continuous $t-$norm. A fuzzy set M in $X^2 \times [0, \infty]$ is called fuzzy $b-$ metric space if, for all $x, y, z \in X$, the following condition hold:*

(1) $M(x, y, 0) = 0$;
(2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, k(t + s))$;
(5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous and* $\lim_{t\to\infty} M(x, y, t) = 1$.

The quadruple $(X, M, *, k)$ is said to be fuzzy $b-$metric space.

Definition 2.5. *Let $(X, M, *)$ be a fuzzy metric space. M is said to be strong if it satisfies additional conditions $M(x, z, t) \geq M(x, y, t) * M(y, z, t)\ \forall\, x, y, z \in X\ \forall t > 0$.*

Definition 2.6.

(1) *Let $(X, M, *)$ be a fuzzy metric space. A sequence x_n in X Said to be convergent to a point $x \in X$ in $(X, M, *)$, if $\lim_{n\to\infty} M(x, y, t) = 1$, $\forall t > 0$.*
(2) *A sequence x_n in X is called a Cauchy sequence in $(X, M, *)$, if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$, for $n, m \in (0, 1)$.*

(3) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 2.7. *(Banach fixed point theorem) Let (X, d) be a non empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed point x^* in X.((i.e) $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, then $\{x_n\} \to x^*$.*

3. Examples

Example 1. *Let $M_d : X \times X \times [0, \infty] \to [0, 1]$, define $M_d(x, y, t) = \frac{t}{t+d(x,y)}$, where d be a $b-$metric space and $a * c = \min\{a, c\}$, for all $a, c \in [0, 1]$. Then show that $(X, M_d, *, k)$ is a fuzzy $b-$metric space.* $M_d(x, y, 0) = 0$, $M_d(x, y, t) = 1 = \frac{t}{t+d(x,y)}$ *which implies that* $d(x, y) = 0$ *implies that* $x = y$, $M_d(x, y, t) = \frac{t}{t+d(x,y)} = \frac{t}{t+d(y,x)} = M_d(y, x, t)$. *We shall show that* $M_d(x, z, k(t + s)) \geq M_d(x, y, t) * M_d(y, z, s)$. *Let* $x, y, z \in X$ *and* $s, t > 0$. *with out loss of generality we assume that* $M_d(x, y, t) \leq M_d(y, z, s)$ *which implies that* $td(y, z) \leq sd(x, y)$

$$M_d(x, y, k(t + s)) = \frac{k(t + s))}{k(t + s) + d(x, z)} \geq \frac{k(t + s)}{k(t + s) + k[d(x, y) + d(y, z)]}$$
$$= \frac{(t + s)}{(t + s) + [d(x, y) + d(y, z)]} = \frac{(t + s)}{(t + s) + d(x, y)} * \frac{(t + s))}{(t + s) + d(y, z)}$$
$$\geq \min\left\{\frac{t}{t + d(x, y)}, \frac{s}{s + d(y, z)}\right\},$$

which implies that $\frac{(t+s)}{(t+s)+[d(x,y)+d(y,z)]} \geq \frac{t}{t+d(x,y)}$ *from this equation we obtained* $td(y, z) \leq sd(x, y)$ *which is true. Hence* $M_d(x, z, k(t+s)) \geq M_d(x, y, t)*M_d(y, z, s)$. $\lim_{t\to\infty} M_d(x, y, t) = 1$ *implies that* $\lim_{t\to\infty} \frac{t}{t+d(x,y)} = 1$ *and* $M_d(x, y, t)$ *is continuous.Therefore* $(X, M_d, *, k)$ *is a fuzzy* $b-$ *metric space.*

Example 2. *Let $M_d : X \times X \times [0, \infty] \to [0, 1]$, define $M_d(x, y, t) = e^{\frac{-d(x,y)}{t}}$, where d be a $b-$metric space and $a * c = \min\{a, c\}$, for all $a, c \in [0, 1]$. Then show that $(X, M_d, *, k)$ is a fuzzy $b-$metric space.* $M_d(x, y, 0) = 0$,$M_d(x, y, t) = 1 = e^{\frac{-d(x,y)}{t}}$ *which implies that* $d(x, y) = 0$ *implies that* $x = y$ $M_d(x, y, t) = e^{\frac{-d(x,y)}{t}} = e^{\frac{-d(y,x)}{t}} = M_d(y, x, t)$.*We shall show that* $M_d(x, z, k(t + s)) \geq M_d(x, y, t) * M_d(y, z, s)$. *Let* $x, y, z \in X$ *and* $s, t > 0$. *With out loss of generality we assume that* $M(x, y, t) \leq$

$M_d(y,z,s)$. *(i.e)* $e^{\frac{-d(x,y)}{t}} \leq e^{\frac{-d(y,z)}{s}}$ *this implies that* $sd(x,y) \geq td(y,z)$ *hence we will obtain*

$$M_d(x,z,k(t+s)) = e^{\frac{-d(x,z)}{k(t+s)}} = e^{\frac{-k(d(x,y)+d(y,z))}{k(t+s)}} = e^{\frac{-d(x,y)}{(t+s)}} * e^{\frac{-d(x,z)}{(t+s)}}$$
$$= \min\left\{e^{\frac{-d(x,y)}{t}}, e^{\frac{-d(y,z)}{s}}\right\} \geq e^{\frac{-d(x,y)}{t}},$$

i.e., $e^{\frac{-(d(x,y)+d(y,z))}{(t+s)}} \geq e^{\frac{-d(x,y)}{t}}$. *Now equation (5) implies that* $td(y,z) \leq sd(x,y)$ *which satisfied. Therefore* $M_d(x,z,k(t+s)) \geq M_d(x,y,t) * M_d(y,z,s)$, $\lim_{t\to\infty} M_d(x, y,t) = 1$ *implies that* $\lim_{t\to\infty} e^{\frac{-d(x,y)}{t}} = 1$ *and* $M_d(x,y,t)$ *is continuous. Therefore* $(X, M_d, *, k)$ *is a fuzzy* $b-$ *metric space.*

Proposition 3.1. *Let* $(X, M, *)$ *be complete fuzzy metric space and* $T : X \to X$ *is continuous function and satisfying the condition*

$$M(Tx,Ty,t) \geq \min\{M(x,Tx,t), M(y,Ty,t), M(x,y,t)\}. \tag{3.1}$$

Moreover the fuzzy metric $M(x,y,t)$ *satisfies the condition* $\lim_{n\to\infty} M(x,y,t) = 1$, *where* $x, y \in X$ *and* $x \neq y$ *then* T *has a unique fixed point in* X.

Proof. Take $\{x_n\}$ be a sequence in X such that $x_{n+1} = Tx_n$. If $x_{n+1} = x_n$ then $Tx_{n+1} = Tx_n = x_n$. This implies that $\{x_n\}$ is a fixed point of X. Suppose that $x_{n+1} \neq x_n$, To prove that $\{x_n\}$ is a Cauchy sequence in X. Put $x = x_{n-1}, y = x_n$, we get

$$\begin{aligned} M(Tx_{n-1}, Tx_n, t) \leq & \min\{M(x_{n-1}, Tx_{n-1}, t), M(x_n, Tx_n, t), \\ & M(x_{n-1}, x_n, t)\} \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Rightarrow M(x_n, x_{n+1}, t) &\geq M(x_{n-1}, x_n, t) \\ \Rightarrow M(x_{n+1}, x_n, t) &\geq M(x_n, x_{n-1}, t). \end{aligned}$$

for all n. Now we put $x = x_{n-1}, y = x_{n-2}$ in (1) we get

$$\begin{aligned} M(Tx_{n-1}, Tx_{n-2}, t) \leq & \min\{M(x_{n-1}, Tx_{n-1}, t), M(x_{n-2}, Tx_{n-2}, t), \\ & M(x_{n-1}, x_{n-2}, t)\} \\ = & \min\{M(x_{n-1}, x_{n-2}, t), M(x_{n-2}, x_{n+1}, t), \\ & M(x_{n-1}, x_{n-2}, t)\} \\ M(x_n, x_{n-1}, t) \geq & M(x_{n-1}, x_{n-2}, t) \end{aligned}$$

$$M\left(x_{n-1}, x_n, t\right) \geq M\left(x_{n-2}, x_{n-1}, t\right), \tag{3.3}$$

for all n. Let as assume that x_n is not a cauchy sequence in X. Then for $0 < \epsilon < 1$, $t > 0$, there exists subsequence $\{x_{n_k}\}$, and $\{x_{m_k}\}$, where $n_k, m_k \geq n$ and $n_k, m_k \in \mathbb{N}$ $(n_k > m_k)$,

$$M\left(x_{n_k}, x_{m_k}, t\right) \leq 1-\epsilon, M\left(x_{n_{k-1}}, x_{m_{k-1}}, t\right) > 1-\epsilon \tag{3.4}$$

and

$$M\left(x_{n_{k-1}}, x_{m_k}, t\right) > 1-\epsilon,$$

$$\begin{aligned}
1-\epsilon &\geq M\left(x_{n_k}, x_{m_k}, t\right) \\
&\geq \min\left\{M\left(x_{n_k}, x_{n_{k-1}}, \frac{t}{2}\right), M\left(x_{n_{k-1}}, x_{m_k}, \frac{t}{2}\right)\right\} \\
&\geq \min\left\{M\left(x_{n_k}, x_{n_{k-2}}, \frac{t}{4}\right), M\left(x_{n_{k-2}}, x_{n_{k-1}}, \frac{t}{4}\right), M\left(x_{n_{k-1}}, x_{m_k}, \frac{t}{2}\right)\right\} \\
&\geq \min\left\{M\left(x_{n_k}, x_{n_{k-2}}, \frac{t}{4}\right), M\left(x_{n_{k-1}}, x_{m_k}, \frac{t}{2}\right)\right\} \\
&\geq \min\left\{M\left(x_{n_{k-1}}, x_{n_{k-1}}, \frac{t}{4}\right), M\left(x_{n_{k-1}}, x_{m_k}, \frac{t}{2}\right)\right\} \\
&\geq \min\left\{1, M\left(x_{n_{k-1}}, x_{m_k}, \frac{t}{2}\right)\right\} \\
&= M\left(x_{n_{k-1}}, x_{m_k}, \frac{t}{2}\right) \\
&> 1-\epsilon,
\end{aligned}$$

which is contradiction. Therefore $\{x_n\}$ is a cauchy sequence in X. Since X is complete , there exists an element $z \in X$ such that $\lim_{n\to\infty} x_n = z$. That is $M\left(x_n, z, t\right) = 1$ as $n \to \infty$. Next prove that limit of z is unique.

Suppose that $\lim_{n\to\infty} u_n = w$, for some $w \in X$, $w \neq z$ then, $M(w, z, t) \geq M(w, u_n, \frac{t}{2}) * M(u_n, z, \frac{t}{2})$,$(a * b = \min(a, b))$. Taking $n \to \infty$, $M(w, z, t) \geq M(w, w, \frac{t}{2}) * M(z, z, \frac{t}{2})$, $M(w, z, t) \geq 1$, which is contradiction. Therefore limit z is unique. To show that z is a fixed point. T is continuous so $u_n \to z \Rightarrow Tu_n \to Tz$.

Consider

$$\begin{aligned}
M(u_n, u_{n+1}, t) &\geq M(u_n, u_{n-1}, t) \\
M(u_n, Tu_n, t) &\geq M(u_n, u_{n-1}, t).
\end{aligned}$$

Taking $n \to \infty$, we get,

$$\begin{aligned} M(z,Tz,T) &\geq M(z,z,t) \\ M(z,z,t) &\geq 1. \end{aligned}$$

Hence $M(Z,Tz,t) = 1$ implies that $Tz = z$. Thus z is a fixed point of T.

To proof uniqueness: Suppose q is a another point of T that is $Tq = q$; $q \neq z$. Now we show that $q = z$:

$$\begin{aligned} 1 &> M(z,q,t) \\ &\geq \min\left\{M(z,z,\frac{t}{2}), M(z,q,\frac{t}{2})\right\} \\ &\geq \min\left\{1, M(z,z,\frac{t}{4}), M(z,q,\frac{t}{4})\right\} \\ &\geq \min\left\{1,1, M(z,z,\frac{t}{8}), M(z,q,\frac{t}{8})\right\} \\ &\geq \min\left\{1,1,1, M(z,z,\frac{t}{16}), M(z,q,\frac{t}{16})\right\} \\ &\geq \cdots \\ &\geq \cdots \\ &\geq \min\left\{1,1,1,\ldots, M(z,q,\frac{t}{2^k})\right\} \\ &= 1 \; as \; k \to \infty, \end{aligned}$$

which is contradiction. Therefore $q = z$. (i.e) z is a fixed point of T. □

ACKNOWLEDGMENT

I thank my supervisor Dr.K.Kannan, Department of Mathematics and Statistics, University of Jaffna, SriLanka for providing me important references to the literature.

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